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KÄHLER CLASSES ON UNIVERSAL MODULI SPACES AND VOLUMINA OF QUOT SPACES

CHRISTIAN OKONEK AND ANDREI TELEMAN

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Contents

1. Introduction 1
2. General theory 7
2.1. Donaldson's quotient connection 7
2.2. A pushforward formula in equivariant cohomology 9
2.3. Sections in Hamiltonian fibre bundles 11
2.4. Families of connections and sections 14
3. Applications 17
3.1. Kähler forms of canonical Hermitian metrics and tautological classes 17
3.2. Volumina of Abelian Quot spaces 24
3.3. Volumina of non-Abelian Quot spaces 29
References 35

1. Introduction

Let (X, g) be compact Hermitian manifold and E a $\mathcal{C}^\infty$ Hermitian bundle on X. Recall that a Hermitian connection A on E is called Hermite-Einstein if its curvature F_A is of type (1,1) and satisfies the Hermite-Einstein equation

$$i\Lambda_g F_A = c\,\mathrm{id}_E \ , \tag{1}$$

for a constant $c \in \mathbb{R}$.

The classical version of the Kobayashi-Hitchin correspondence states that a holomorphic bundle $\mathcal{E}$ over a compact Gauduchon manifold (X, g) is stable if and only if it admits a Hermitian metric h such that the associated Chern connection $A_{\mathcal{E},h}$ is Hermite-Einstein and irreducible. This statement has first been proved by Donaldson [Do1] for projective-algebraic surfaces and for projective manifolds endowed with Hodge metrics [Do2], later by Uhlenbeck and Yau for Kähler manifolds [UY1], [UY2], and finally by Li und Yau for arbitrary compact complex manifolds endowed with Gauduchon metrics ([LY], [LT1]).

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This statement, which gives a differential geometric characterization of stability, can be interpreted as an isomorphism between moduli spaces; more precisely, it yields a real analytic isomorphism

$$\mathcal{M}^{\mathrm{HE}}(E)^* \xrightarrow{KH} \mathcal{M}^{\mathrm{st}}(E) \ .$$

Here $\mathcal{M}^{\mathrm{HE}}(E)^*$ is the moduli space of irreducible Hermite-Einstein connections on E, and $\mathcal{M}^{\mathrm{st}}(E)$ denotes the moduli space of stable holomorphic structures on E. This geometric interpretation of the original Kobayashi-Hitchin correspondence found important applications in differential topology: it allowed Donaldson to describe explicitly certain instanton moduli spaces on algebraic surfaces and to compute the first Donaldson invariants. A second fundamental application of the classical Kobayashi-Hitchin correspondence was Donaldson's non-vanishing theorem for his polynomial invariants on algebraic surfaces. This result was based on the fundamental fact that the Kähler class of the canonical metric of the moduli space can be identified with a tautological cohomology class [DK]. One of the motivations of this article is to show that this phenomenon holds in great generality.

An important generalization of the classical Kobayashi-Hitchin correspondence concerns moduli spaces of pairs consisting of oriented (see [LT2] for the terminology) connections or holomorphic structures coupled with sections in associated bundles. For instance, in the theory of Hitchin pairs [Hi] one considers a differentiable Hermitian vector bundle E, a holomorphic structure $\mathcal{D}$ on $\det(E)$, and a fixed holomorphic bundle $\mathcal{E}_0$ on X. The complex geometric problem in this case is the classification of pairs $(\mathcal{E},\varphi)$ consisting of a holomorphic structure $\mathcal{E}$ on E inducing $\mathcal{D}$ on $\det(E)$, and a section $\varphi \in H^0(\mathcal{E}nd(\mathcal{E})\otimes\mathcal{E}_0)$, modulo the complex gauge group $\mathcal{G}^{\mathbb{C}} = \Gamma(X,\mathrm{SL}(E))$. In the classical case one takes $\mathcal{E}_0 := \Omega^1_X$ [Hi], [Si].

Let a be the Chern connection of the holomorphic structure $\mathcal{D}$ on the Hermitian line bundle $\det(E)$. The corresponding gauge theoretical problem concerns the moduli space of pairs (A,φ) consisting of a Hermitian connection A on E inducing a on $\det(E)$ and a section $\varphi \in A^0(\mathrm{End}(E)\otimes\mathcal{E}_0)$ such that the following Hermite-Einstein type equation is satisfied:

$$i\Lambda F^0_A + \frac{1}{2}[\varphi,\varphi^{*_h}] = 0 \ , \ F^{0,2}_A = 0$$

Defining stability of Hitchin pairs in an appropriate way one obtains again a Kobayashi-Hitchin correspondence

$$\mathcal{M}^{\mathrm{HE}}_a(E,\mathcal{E}_0) \xrightarrow{KH_{a,\mathcal{E}_0}} \mathcal{M}^{\mathrm{st}}_{\mathcal{D}}(E,\mathcal{E}_0) \ .$$

Many other important gauge theoretical problems fit into this framework, for instance:

(1) Moduli spaces of vortices [Br], [GP], [HL], [Th],
(2) Moduli spaces of oriented pairs [OT1], [Te],
(3) Moduli spaces of Witten triples [W], [Bi], [Dü].

All these Kobayashi-Hitchin type correspondences can be interpreted, at least at a formal level, as infinite dimensional versions of the fundamental isomorphism between symplectic quotients (defined by a moment map) and GIT quotients (corresponding to a suitable stability condition) [Ki].

The idea of a universal Kobayashi-Hitchin correspondence, which specializes to all these isomorphisms of moduli spaces is therefore very natural and important. General versions have been obtained by Banfield [Ban] for connections in principal bundles over Kähler manifolds coupled with sections in associated vector bundles, and by Mundet i Riera [Mu] for connections in principal bundles over Kähler manifolds coupled with sections in associated Kählerian fibre bundles. The universal Kobayashi-Hitchin correspondence of [LT2] deals with (oriented) pairs on Gauduchon manifolds, and identifies the complex geometric stability concept corresponding to the gauge theoretical equations. The final result is a universal isomorphism $KH : \mathcal{M}^* \to \mathcal{M}^{\rm st}$ between a gauge theoretic moduli space $\mathcal{M}^*$ of irreducible (oriented) Hermite-Einstein pairs and a complex geometric moduli space $\mathcal{M}^{\rm st}$ of stable (oriented) pairs.

In this very general framework it is shown that the moduli space $\mathcal{M}^*$ comes with a canonical metric, which is Hermitian with respect to the complex structure induced by the Kobayashi-Hitchin isomorphism KH and is strongly KT, i.e., its Kähler form Ω satisfies the equation $\partial\bar\partial\Omega = 0$. The idea of the proof is to show that Ω can be written as a sum $\Omega = \Omega_1 + \Omega_2$ of two terms, each term Ω_i being obtained by fibre integration. More precisely

$$\Omega_1 = p_{\mathcal{M}^*}(\alpha_1 \wedge p_X^*(\omega_g^{n-1}))\ ,\ \Omega_2 = (n-1)! p_{\mathcal{M}^*}(\alpha_2 \wedge p_X^*(\mathrm{vol}_g))\ ,$$

where α_1 and α_2 are *closed* forms on $\mathcal{M}^* \times X$. Note that α_1 is a characteristic form of the universal bundle $\mathbb{P}$ on $\mathcal{M}^* \times X$, hence the first term Ω_1 is constructed in the same way as the de Rham representatives of Donaldson's tautological classes [DK]. The first goal of this article is to prove that, under very general assumptions, the closed form α_2 is also a characteristic form of $\mathbb{P}$ and to identify the corresponding characteristic class explicitly. This is a highly non-trivial result whose proof is based on a push-forward formula in equivariant cohomology. A similar, but less general result, has been obtained with different methods by Baptista [Bap].

Therefore, assuming that $d(\omega_g^{n-1}) = 0$ (i.e., g is semi-Kähler) one obtains a canonical Kähler metric on $\mathcal{M}^*$ whose Kähler form is the de Rham representative of the sum of two Donaldson type classes. This fundamental result can be regarded as a universal generalization of formulae obtained by Manton–Nasir [MN] and Perutz [P] in the case of vortices on Riemann surfaces, and by Donaldson in the case of instantons on algebraic surfaces [DK].

In order to formulate our main results more precisely, we recall briefly the formalism introduced in [LT2] in the special case of Kähler manifolds, which suffices for our purposes.

Suppose that (X, J, g) is an n-dimensional Kähler manifold and $\pi : P \to X$ is a principal K-bundle on X, with K a compact Lie group. We fix an epimorphism $r : K \to K_0$ of compact Lie groups, and denote by N its kernel. Let $P_0 := P \times_K K_0$ be the associated K_0-bundle and fix a connection $A_0 \in \mathcal{A}(P_0)$ of type $(1, 1)$.

Suppose we are given a left K-action on a Kähler manifold (F, J_F, g_F) by holomorphic isometries, and a moment map μ_F for this action on the corresponding symplectic manifold (F, ω_F). We fix a K-invariant inner product k on $\mathfrak{k}$ and we denote by $m_F : F \to \mathfrak{k}$ the $\mathfrak{k}$-valued map defined by $\mu_F = k(m_F, \cdot)$. Let $p_\mathfrak{n} : \mathfrak{k} \to \mathfrak{n}$ be the k-orthogonal projection onto the Lie algebra $\mathfrak{n}$ of N. The symmetric bilinear form $h : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ defined by

$$h(x, y) := k(p_\mathfrak{n}(x), p_\mathfrak{n}(y))$$

is K-invariant and non-degenerate on $\mathfrak{n}$.

Let $E := P \times_K F$ be the associated bundle. The gauge group

$$\mathcal{G} := \Gamma(X, P \times_K N)$$

acts from the left on the space of A_0-oriented pairs $\mathcal{A}_{A_0}(P) \times \Gamma(X, E)$. This space has a natural Kähler metric depending on the triple (k, g, g_F), whose Kähler form is given by the formula ([LT2] p. 66):

$$\Omega((\alpha,\psi),(\beta,\chi)) := \int_X h(\alpha \wedge \beta) \wedge \omega_g^{n-1} + (n-1)! \int_X \omega_F(\psi,\chi)\mathrm{vol}_g$$

The fundamental object associated with these data is the moduli space $\mathcal{M}^*$ of equivalence classes of irreducible oriented pairs $(A, \varphi) \in \mathcal{A}_{A_0}(P) \times \Gamma(X, E)$ satisfying the generalized vortex equations

$$\left\{\begin{array}{ccc} F_A^{02} & = & 0 \\ \varphi \text{ is } A\text{-holomorphic} & & \\ p_{\mathfrak{n}}\left[\Lambda_g F_A + m_F(\varphi)\right] & = & 0\ . \end{array}\right. \qquad (V)$$

In the third equation we used the same symbol for the vector bundle epimorphism $P \times_K \mathfrak{k} \to P \times_K \mathfrak{n}$ induced by $p_{\mathfrak{n}} : \mathfrak{k} \to \mathfrak{n}$.

Denote by $\omega_{\mathcal{M}^*}$ the Kähler form of the canonical Kähler metric induced on the moduli space $\mathcal{M}^*$ of irreducible A_0-oriented pairs. Using a standard construction one defines a principal K-bundle $\mathbb{P}$ over $\mathcal{M}^* \times X$ and a tautological K-equivariant map $\Phi : \mathbb{P} \to F$, which can be regarded as a universal section in the associated F-bundle. Our main results are:

Theorem 3.1. *The canonical Hermitian metric on $\mathcal{M}^*$ is Kähler and its Kähler form is*

$$\omega_{\mathcal{M}^*} = p_{\mathcal{M}^*,*}\left[-\frac{1}{2}[h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}})] \wedge p_X^*(\omega_g^{n-1}) + (n-1)!\eta \wedge p_X^*(\mathrm{vol}_g)\right]\ ,$$

where $\Omega_{\mathbb{A}}$ denotes the curvature of the universal connection, and η is the horizontal part of $\Phi^*(\omega_F) - \langle \mu_F \circ \Phi, \Omega_{\mathbb{A}}\rangle$; η is a closed 2-form. If μ_F can be written as $\mu_F = \mu_0 + \tau$, where μ_0 is an exact moment map and $\tau \in \mathfrak{k}^\vee$ is K-invariant, then the de Rham cohomology class $[\omega_{\mathcal{M}^*}]_{\mathrm{DR}}$ is given by

$$[\omega_{\mathcal{M}^*}]_{\mathrm{DR}} = p_{\mathcal{M}^*,*}\left[-\frac{1}{2}[h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}})] \cup p_X^*[\omega_g]^{n-1} - (n-1)![\tau(\Omega_{\mathbb{A}})] \cup p_X^*[\mathrm{vol}_g]\right]\ .$$

For the linear case we obtain a more precise result. If the chosen unitary representation $\rho : K \to \mathrm{U}(F)$ satisfies a technical condition (similar to a condition in [Bap]), then one obtains an equality between forms (not just de Rham classes).

Theorem 3.2. *Let F be a Hermitian vector space, and $\rho : K \to \mathrm{U}(F)$ a unitary representation. For a K-invariant linear form $\tau \in \mathfrak{k}^\vee$ put $\mu_F := \mu_0 + \tau$, where μ_0 is the standard moment map for the K-action on F.*

Suppose there exists a K-invariant element $a_0 \in \mathfrak{n}$ such that $\rho_*(a_0) = i\mathrm{id}_F$. Then the Kähler form $\omega_{\mathcal{M}^*}$ is given by

$$\omega_{\mathcal{M}^*} = p_{\mathcal{M}^*,*}\left[-\frac{1}{2}h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}}) \wedge p_X^*(\omega_g)^{n-1} - (n-1)!\tau(\Omega_{\mathbb{A}}) \wedge p_X^*(\mathrm{vol}_g)\right]\ ,$$

hence it coincides with the fibre integration of a characteristic form of the connection $\mathbb{A}$ on the universal bundle $\mathbb{P}$.

In the special case of Higgs bundles similar fibre integration formulae have been obtained in [BiSc].

The second goal of our article is to use these theoretical results to compute the volumina of certain interesting moduli spaces with respect to the canonical metrics induced by the Kobayashi-Hitchin correspondence.

The computation of these volumina is not only an interesting mathematical problem, but is also important from a physical point of view. The first computation of such volumina is due to Manton and Nasir [MN]. These authors consider the vortex equation with fixed parameter $t = \frac{1}{2}$ on a Riemann surface (X, g) of genus $\mathfrak{g}$. Assuming $\mathrm{Vol}_g(X) > 4\pi d$ the symmetric power $X^{(d)}$ can be identified with the moduli spaces of vortices of degree d. Their main result is the computation of the volumina of $X^{(d)}$ with respect to the induced metric (see [MN], (3.22)):

$$\mathrm{Vol}_\omega(X^{(d)}) = \sum_{i=0}^{\min(d,\mathfrak{g})} (4\pi)^i \binom{\mathfrak{g}}{i} \frac{1}{(d-i)!} (\mathrm{Vol}_g(X) - 4\pi d)^{d-i} \ .$$

A partial generalization of this result is due to Baptista, who computed the volumina of the moduli spaces of semi-local Abelian vortices when the degree d is larger than $2\mathfrak{g} - 2$ (see [Bap] (52)).

Both formulae are (up to normalization factors) special cases of the formulae which we will prove in this article (see section 3):

Let X be a compact complex manifold of dimension n, $\mathcal{E}_0$ a locally free sheaf of rank r_0 on X, and E a differentiable vector bundle on X. We denote by $\mathcal{Q}uot^E_{\mathcal{E}_0}$ the Quot space of equivalence classes of quotients $q : \mathcal{E}_0 \to \mathcal{Q}$ with locally free kernel of differentiable type E. When E is of rank 1 and $m := c_1(E) \in \mathrm{NS}(X)$ we put $\mathcal{Q}uot^m_{\mathcal{E}_0} = \mathcal{Q}uot^E_{\mathcal{E}_0}$.

We fix a Hermitian metric h_0 on $\mathcal{E}_0$ and denote by A_0 the associated Chern connection. Let g be a Kähler metric on X, $t \in \mathbb{R}$ a parameter and put $\mathfrak{t} := \frac{(n-1)!\mathrm{Vol}_g(X)}{2\pi} t$. The moduli space $\mathcal{M}^*_t(E, A_0)$ of irreducible t-vortices of type (E, A_0) can be identified via the Kobayashi-Hitchin correspondence with the moduli space $\mathcal{M}^{\mathrm{st}}_{\mathfrak{t}}(E, \mathcal{E}_0)$ of $\mathfrak{t}$-stable pairs of type $(E, \mathcal{E}_0)$, and for sufficiently large $\mathfrak{t}$ the latter can be identified with the Quot space $\mathcal{Q}uot^E_{\mathcal{E}_0}$ [OT2]. In this way we obtain natural metrics on $\mathcal{Q}uot^E_{\mathcal{E}_0}$.

Note that our moduli spaces come with a natural symmetry, since any morphism of complex Lie groups $G \to \mathrm{Aut}(\mathcal{E}_0)$ induces a natural holomorphic G-action on the moduli spaces $\mathcal{M}^{\mathrm{st}}_{\mathfrak{t}}(E, \mathcal{E}_0)$ and $\mathcal{Q}uot^E_{\mathcal{E}_0}$. Our formalism yields a natural G-equivariant lift

$$[\omega_t]^G \in H^2_G(\mathcal{M}^{\mathrm{st}}_{\mathfrak{t}}(E, \mathcal{E}_0), \mathbb{R})$$

of the Kähler class $[\omega_t]$ in terms of equivariant Chern classes of the universal bundle.

Theorem 3.6. *Let $\mathscr{E}$ be the universal bundle on $\mathcal{M}^{\mathrm{st}}_{\mathfrak{t}}(E, \mathcal{E}_0) \times X$. For any morphism of Lie groups $G \to \mathrm{Aut}(\mathcal{E}_0)$, the class*

$$\left\{ -4\pi^2 \mathrm{ch}^G_2(\mathscr{E}) \cup p^*_X[\omega^{n-1}_g] - 2t\pi(n-1)! c^G_1(\mathscr{E}) \cup p^*_X[\mathrm{vol}_g] \right\} / [X]$$

is a lift of the Kähler class $[\omega_t]$ to $H^2_G(\mathcal{M}^{\mathrm{st}}_{\mathfrak{t}}(E, \mathcal{E}_0), \mathbb{R})$.

This general result is very useful for explicit computations of volumina of moduli spaces using localization methods.

In the case when E is a line bundle with $c_1(E)=m$ we have a natural identification $\mathcal{M}^{\rm st}_{\mathfrak{t}}(E,\mathcal{E}_0)=\mathcal{Q}uot^E_{\mathcal{E}_0}$ for any $\mathfrak{t}>-\deg(E)$, and

$$\frac{1}{4\pi^2}[\omega_{\mathcal{M}^*_t}]=\theta+(\deg(E)+\mathfrak{t})\gamma\ , \tag{2}$$

where θ is the pull-pack of the theta class of $\mathrm{Pic}^m(X)$, and $\gamma=-c_1(\mathscr{E}\big|_{\mathcal{Q}uot^E_{\mathcal{E}_0}\times\{x_0\}})$.

Formula (2) specializes to a result obtained by Perutz for $r_0=1$ and $n=1$ (see [P] Theorem 3).

When X is a Riemann surface and $r_0=1$, the Quot space $\mathcal{Q}uot^m_{\mathcal{E}_0}$ can be identified with the symmetric power $X^{(d)}$, $d:=\deg(\mathcal{E}_0)-\deg(E)$. Using known identities for the intersection numbers $\langle\theta^i\gamma^j,[X^{(d)}]\rangle$ we obtain

$$\mathrm{Vol}_{\omega_t}(\mathcal{Q}uot^E_{\mathcal{E}_0})=(4\pi^2)^d\sum_{i=0}^{\min(d,\mathfrak{g})}\binom{\mathfrak{g}}{i}\frac{1}{(d-i)!}(\deg(E)+\mathfrak{t})^{d-i}\ ,$$

which (up to the normalization factor π^d) specializes to the Manton-Nasir formula.

Our second application concerns the volumina of the Abelian Quot spaces $\mathcal{Q}uot^E_{\mathcal{E}_0}$ for arbitrary $r_0\geq 1$, $n\geq 1$. We will say that a pair $(m,\mathcal{E}_0)$ is acyclic if $H^i(\mathcal{L}^\vee\otimes\mathcal{E}_0)=0$ for all $i>0$ and all $[\mathcal{L}]\in\mathrm{Pic}^m(X)$. If this is the case then $\mathcal{Q}uot^E_{\mathcal{E}_0}$ is a projective bundle over $\mathrm{Pic}^m(X)$, and we obtain an explicit formula for its volume:

$$\mathrm{Vol}_{\omega_t}(\mathcal{Q}uot^E_{\mathcal{E}_0})=\frac{(4\pi^2)^N}{N!}\Bigg\{\Bigg(\sum_{k=R-1}^{N}\binom{N}{k}(\deg_g(E)+\mathfrak{t})^{N-k}\theta^k\Bigg)\wedge$$

$$\wedge\exp\left(\sum_{i=1}^{q}\sum_{s=0}^{n-i}\frac{(-1)^s}{is!}\mathfrak{k}_{m^sC_{n-i-s}}\right)\Bigg\}(h_1,\dots,h_{2q})\ . \tag{3}$$

In this formula $(h_1,\dots,h_{2q})$ is a basis of $H^1(X,\mathbb{Z})$ compatible with its natural orientation, $R:=\chi(\mathcal{L}^\vee\otimes\mathcal{E}_0)$, and $N:=R+q(X)-1$ is the dimension of $\mathcal{Q}uot^E_{\mathcal{E}_0}$. The classes $C_i\in H^{2i}(X,\mathbb{Q})$ are defined by the identity $\mathrm{ch}(\mathcal{E}_0)\mathrm{td}(X)=\sum_i C_i$, and for a class $\mathfrak{c}\in H^{2n-k}(X,\mathbb{Q})$ we denote by $\kappa_{\mathfrak{c}}\in\mathrm{Alt}^k(H^1(X,\mathbb{Z}),\mathbb{Q})$ the form defined by

$$\mathfrak{k}_{\mathfrak{c}}(x_1,\dots,x_k):=\langle x_1\cup\dots\cup x_k\cup\mathfrak{c},[X]\rangle\ .$$

In the special case when $n=1$ and $\mathcal{E}_0=\mathcal{O}_X^{\oplus r_0}$, formula (3) specializes to Baptista's result mentioned above.

The last section deals with the computation of volumina of non-Abelian Quot spaces $\mathcal{Q}uot^E_{\mathcal{E}_0}$ on Riemann surfaces in the case $r=r_0$. These moduli spaces are always smooth and projective, but their cohomology algebra is unknown. Since for $\mathcal{E}_0=\mathcal{O}_X^{\oplus r_0}$ one obtains precisely Weil's moduli spaces of matrix divisors, we call them moduli spaces of *twisted matrix divisors*. For fixed data (E,t,g) the volume of $\mathcal{Q}uot^E_{\mathcal{E}_0}$ depends only on the topological type of $\mathcal{E}_0$, hence we can suppose that $\mathcal{E}_0=\bigoplus_{i=1}^{r_0}\mathcal{L}_i$ splits as a direct sum of holomorphic line bundles $\mathcal{L}_i$. In this case one obtains a natural action of $[\mathbb{C}^*]^{r_0}$ on $\mathcal{Q}uot^E_{\mathcal{E}_0}$ and the computation of the volume of this moduli space can be reduced – using localization techniques – to an integration over the fixed point locus of this action. For our computations we will use the $\mathbb{C}^*$-action on $\mathcal{Q}uot^E_{\mathcal{E}_0}$ induced by the morphism $\mathbb{C}^*\to[\mathbb{C}^*]^{r_0}$ associated with a system

$(w_1, \dots, w_r)$ of pairwise distinct weights. With this choice we get

$$[\mathcal{Q}uot^E_{\mathcal{E}_0}]^{\mathbb{C}^*} = \coprod_{\underline{d}\in I_r(d)} \prod_{i=1}^r \mathcal{Q}uot^{l_i-d_i}_{\mathcal{L}_i} \ ,$$

where $d := \deg(\mathcal{E}_0) - \deg(\mathcal{E})$, $l_i := \deg(\mathcal{L}_i)$ and

$$I_r(d) := \{\underline{d} = (d_1, \dots, d_r) \in \mathbb{N}^r | \ \sum_{i=1}^r d_i = d\} \ .$$

Using Theorem 3.6 we obtain an explicit expression for the equivariant restrictions

$$[\omega_t]^{\mathbb{C}^*}\Big|_{\prod_{i=1}^r \mathcal{Q}uot^{l_i-d_i}_{\mathcal{L}_i}} \ .$$

Applying standard localization techniques, we obtain a complicated general formula for $\mathrm{Vol}_{\omega_t}(\mathcal{Q}uot^E_{\mathcal{E}_0})$, which is not explicit, because it involves the coefficients of the Taylor expansion of a transcendental function in $2r$ variables.

However, our formula leads to an algorithm which can be implemented on a computer. Deriving a general closed formula in terms of $\mathrm{Vol}_g(X)$ and the topological data $(\mathfrak{g}, d, l, r)$ is an interesting combinatorial problem, which we cannot solve at the moment.

Note that the computation of the volume of $\mathcal{Q}uot^E_{\mathcal{E}_0}$ gives a purely algebraic geometric result: The moduli spaces $\mathcal{Q}uot^E_{\mathcal{E}_0}$ of twisted matrix divisors come with natural Grothendieck embeddings

$$j_n : \mathcal{Q}uot^E_{\mathcal{E}_0} \hookrightarrow \mathbb{P}(V_n)$$

in projective spaces (for $n \gg 0$). The Kähler class $c_1(j_n^*(\mathcal{O}_{V_n}(1)))$ can be identified with a multiple of our Kähler class $[\omega_t]$ for a suitable choice of the parameter t, more precisely

$$c_1(j_n^*(\mathcal{O}_{V_n}(1))) = \frac{1}{4\pi^2}\left[\omega_{\frac{2\pi(n-\mathfrak{g})}{\mathrm{Vol}_g(X)}}\right] \ .$$

Therefore our computation of $\mathrm{Vol}_{\omega_t}(\mathcal{Q}uot^E_{\mathcal{E}_0})$ yields in particular the degrees of the images of the Quot spaces $\mathcal{Q}uot^E_{\mathcal{E}_0}$ under the Grothendieck embeddings.

We illustrate all this with explicit formulae in the case $r = 2$ and $d \in \{1, 2\}$.

2. General theory

2.1. **Donaldson's quotient connection.** Suppose we have a G-equivariant principal K-bundle $\pi : P \to M$, where G is a Lie group acting *freely* on M. By definition this means that G acts on P by K-bundle automorphisms covering the action on M. Suppose that the quotient $\bar{M} := M/G$ has a manifold structure such that the projection $q : M \to \bar{M}$ is a submersion. This implies that q is a principal G-bundle over $\bar{M}$.

In this situation the quotient $\bar{P} := P/G$ has a natural structure of a principal K-bundle over $\bar{M}$. This bundle $\bar{\pi} : \bar{P} \to \bar{M}$ will be called the G-quotient of π.

Let A be a G-invariant connection on P. The aim of this appendix is to construct a quotient connection $\bar{A}$ on $\bar{P}$ and to compute its curvature. The construction depends on the choice of an auxiliary connection Γ on the G-bundle $q : M \to \bar{M}$, i.e., a G-invariant distribution on M which is horizontal with respect to the projection $q : M \to \bar{M}$.[1]

The construction of this quotient connection follows the method explained in [DK, section 5.2.3] for linear connections, and later in the general framework of the *universal Kobayashi-Hitchin correspondence* in [LT2, section 6.1.3]. We believe that the general construction of the quotient connection and the computation of its curvature in the general framework of quotient of principal bundles is of independent interest, so we explain it briefly here.

We denote by $\theta_A \in A^1(P, \mathfrak{k})$ the connection form of A and by Γ^A the horizontal distribution associated with A. The intersection $(\pi_*)^{-1}(\Gamma) \cap \Gamma^A$ defines a distribution on P which is invariant with respect to both the K-action and the G-action, and is horizontal with respect to the projection $P \to \bar{M}$. Let $\mathfrak{q} : P \to \bar{P}$ be the natural projection covering $q : M \to \bar{M}$.

Definition 2.1. *The quotient connection $\bar{A}$ on the bundle $\bar{\pi} : \bar{P} \to \bar{M}$ is the connection associated with the distribution $\Gamma^{\bar{A}} := \mathfrak{q}_*(\pi_*^{-1}(\Gamma) \cap \Gamma^A)$.*

The curvature of $\bar{A}$ can be computed as follows: Let $\bar{\xi}$, $\bar{\eta}$ be vector fields on $\bar{M}$ and let ξ, η be their Γ-horizontal lifts on M. The Lie bracket $[\xi, \eta]$ can be decomposed as the sum $[\xi, \eta]^h + [\xi, \eta]^v$ of its Γ-horizontal and Γ-vertical components, and one has

$$[\xi, \eta]^v = \Omega_\Gamma(\xi, \eta)^{\#}_M \ .$$

In this formula we use the notation $a^{\#}_M$ for the vector field on M corresponding to an element $a \in \mathfrak{g}$ via the given (left) action, whereas Ω_Γ is the curvature of the connection defined by Γ in the G-principal bundle $q : M \to \bar{M}$ constructed using the *right* action corresponding to the original left action on M. This explains why there is no $-$ sign in front of the right hand side.

Let $\tilde{\xi}$, $\tilde{\eta}$ be the A-horizontal lifts of ξ, η to P. These vector fields are obviously sections of $\pi_*^{-1}(\Gamma) \cap \Gamma^A$. Let $m \in M$ and let $p \in P_m$ be a lift of m in P. The A-horizontal and A-vertical components of the bracket $[\tilde{\xi}, \tilde{\eta}]$ in p are:

$$[\tilde{\xi}, \tilde{\eta}]^h_p = [\xi, \eta]^{\sim}_p = \{[\xi, \eta]^h_m\}^{\sim}_p + \{[\xi, \eta]^v_m\}^{\sim}_p = \{[\xi, \eta]^h_m\}^{\sim}_p + \{\Omega_\Gamma(\xi_m, \eta_m)^{\#}_{M,m}\}^{\sim}_p \ ,$$

$$[\tilde{\xi}, \tilde{\eta}]^v_p = -\Omega_A(\tilde{\xi}_p, \tilde{\eta}_p)^{\#}_{P,p} \ . \tag{4}$$

On the other hand, since $\pi : P \to M$ is a morphism of left G-manifolds, it follows that for every $\alpha \in \mathfrak{g}$ one has $\pi_*(\alpha^{\#}_{P,p}) = \alpha^{\#}_{M,m}$, so that the A-horizontal component of $\alpha^{\#}_{P,p}$ is $\{\alpha^{\#}_{M,m}\}^{\sim}_p$. The A-vertical component of $\alpha^{\#}_{P,p}$ is $\{\theta_A(\alpha^{\#}_{P,p})\}^{\#}_{P,p}$ by definition of the connection form θ_A. Therefore

$$\{\alpha^{\#}_{M,m}\}^{\sim}_p = \alpha^{\#}_{P,p} - \{\theta_A(\alpha^{\#}_{P,p})\}^{\#}_{P,p} \ .$$

Applying this formula to $\alpha = \Omega_\Gamma(\xi_m, \eta_m)$ and using (4) we get

$$[\tilde{\xi}, \tilde{\eta}]_p = [\tilde{\xi}, \tilde{\eta}]^h_p + [\tilde{\xi}, \tilde{\eta}]^v_p$$

$$= \{[\xi, \eta]^h_m\}^{\sim}_p + \Omega_\Gamma(\xi_m, \eta_m)^{\#}_{P,p} - \{\theta_A(\Omega_\Gamma(\xi_m, \eta_m)^{\#}_{P,p})\}^{\#}_{P,p} - \Omega_A(\tilde{\xi}_p, \tilde{\eta}_p)^{\#}_{P,p} \ .$$

Now we take the image of both sides of this equality via the differential of the projection $\mathfrak{q} : P \to \bar{P}$ at p. Put $\bar{p} := \mathfrak{q}(p)$, and let $\tilde{\bar{\xi}}$, $\tilde{\bar{\eta}}$ be the $\bar{A}$-horizontal lifts of $\bar{\xi}$

[1] Note that G acts on M from the left so, using the standard terminology of [KN], Γ is a connection in the principal G-bundle obtained by endowing M with the right action $m \cdot g := g^{-1} m$.

and $\bar{\eta}$. One easily sees that the push-forwards $\mathfrak{q}_*(\tilde{\xi})$ and $\mathfrak{q}_*(\tilde{\eta})$ are defined and equal to $\tilde{\bar{\xi}}$ and $\tilde{\bar{\eta}}$ respectively. From this one shows that $\mathfrak{q}_*[\tilde{\xi},\tilde{\eta}] = [\tilde{\bar{\xi}},\tilde{\bar{\eta}}]$. On the other hand $\mathfrak{q}_*\left(\Omega_\Gamma(\xi_m,\eta_m)^{\#}_{P,p}\right) = 0$, because the vector field $\Omega_\Gamma(\xi_m,\eta_m)^{\#}_P$ is tangent to the G-orbits. Finally $\mathfrak{q}_*\left(\{[\xi,\eta]^h_m\}^{\sim}_p\right)$ is $\bar{A}$-horizontal, because $\{[\xi,\eta]^h_m\}^{\sim}_p$ belongs to $\pi_*^{-1}(\Gamma)\cap\Gamma^A$. Therefore

$$[\tilde{\bar{\xi}},\tilde{\bar{\eta}}]_{\hat{p}} = \mathfrak{q}_*\left\{\{[\xi,\eta]^h_m\}^{\sim}_p\right\} - \mathfrak{q}_*\left\{\left\{\theta_A(\Omega_\Gamma(\xi_m,\eta_m)^{\#}_{P,p}) + \Omega_A(\tilde{\xi}_p,\tilde{\eta}_p)\right\}^{\#}_{P,p}\right\} .$$

Since $\mathfrak{q}$ is K-equivariant we have

$$\mathfrak{q}_*\left\{\left\{\theta_A(\Omega_\Gamma(\xi_m,\eta_m)^{\#}_{P,p}) + \Omega_A(\tilde{\xi}_p,\tilde{\eta}_p)\right\}^{\#}_{P,p}\right\} = \left\{\theta_A(\Omega_\Gamma(\xi_m,\eta_m)^{\#}_{P,p}) + \Omega_A(\tilde{\xi}_p,\tilde{\eta}_p)\right\}^{\#}_{\bar{P},\bar{p}} .$$

But $\mathfrak{q}_*\left\{\{[\xi,\eta]^h_m\}^{\sim}_p\right\}$ is $\bar{A}$-horizontal and $\Omega_{\bar{A}}(\tilde{\bar{\xi}}_{\bar{p}},\tilde{\bar{\eta}}_{\bar{p}})^{\#} = -[\tilde{\bar{\xi}},\tilde{\bar{\eta}}]^v$. Hence we obtain

$$\left\{\Omega_{\bar{A}}(\tilde{\bar{\xi}}_{\bar{p}},\tilde{\bar{\eta}}_{\bar{p}})\right\}^{\#}_{\bar{P},\bar{p}} = \left\{\Omega_A(\tilde{\xi}_p,\tilde{\eta}_p) + \theta_A(\Omega_\Gamma(\xi_m,\eta_m)^{\#}_{P,p})\right\}^{\#}_{\bar{P},\bar{p}} . \tag{5}$$

This proves

Proposition 2.2. *Let $m\in M$, $p\in P_m$, $\bar{m}:=q(m)$, $\bar{\xi}$, $\bar{\eta}\in T_{\bar{m}}\bar{M}$, ξ, $\eta\in T_mM$ their Γ-horizontal lifts, $\tilde{\xi}$, $\tilde{\eta}\in T_pP$ the A-horizontal lifts of ξ, η, and $\tilde{\bar{\xi}}$, $\tilde{\bar{\eta}}\in T_{\bar{p}}\bar{P}$ the $\bar{A}$-horizontal lifts of $\bar{\xi}$, $\bar{\eta}$ at the point $\bar{p}:=\mathfrak{q}(p)$. The curvature of the quotient connection $\bar{A}$ is given by the formula*

$$\Omega_{\bar{A}}(\tilde{\bar{\xi}},\tilde{\bar{\eta}}) = \Omega_A(\tilde{\xi},\tilde{\eta}) + \theta_A(\Omega_\Gamma(\xi,\eta)^{\#}_{P,p}) . \tag{6}$$

2.2. **A pushforward formula in equivariant cohomology.** Let K be a compact Lie group, and F a differentiable manifold endowed with a left K-action. The Cartan algebra of the K-manifold F (see [GGK] p. 198) is the differential graded $\mathbb{R}$-algebra $A^*_K(F) := \left[\mathbb{R}[\mathfrak{k}]\otimes A^*(F)\right]^K$ of K-invariant $A^*(F)$-valued polynomials on the Lie algebra $\mathfrak{k}$; the grading is given by $\deg(f\otimes\eta) := 2\deg(f)+\deg(\eta)$, and the differential is

$$d_K(\alpha)(a) := d(\alpha(a)) + \iota_{a^{\#}}(\alpha(a)) , \tag{7}$$

for any $\alpha\in\mathbb{R}[\mathfrak{k}]\otimes A^*(F)$ regarded as a polynomial map $\mathfrak{k}\to A^*(F)$. Here $a\in\mathfrak{k}$ and $a^{\#}$ stands for the vector field on F associated with a. The cohomology $H^*_K(F)$ of this algebra is canonically isomorphic to the real equivariant cohomology $H^*_K(F,\mathbb{R})$. A differentiable equivariant map $F\to F'$ defines a morphism of differential graded $\mathbb{R}$-algebras $\varphi^*_K : A^*_K(F')\to A^*_K(F)$ (given by pullback of forms), and the assignment $F\to A^*_K(F)$ defines a contravariant functor on the category of K-manifolds with values in the category of differential graded $\mathbb{R}$-algebras.

In order to extend the functoriality of the assignment $F\to A^*_K(F)$ to right K-manifolds and equivariant maps between a *right* and a left K-manifold, one introduces the Cartan algebra associated with a right K-manifold P by endowing the graded $\mathbb{R}$-algebra $A^*_K(P) := \left[\mathbb{R}[\mathfrak{k}]\otimes A^*(P)\right]^K$ with the differential

$$d_K(\alpha)(a) := d(\alpha(a)) - \iota_{a^{\#}}(\alpha(a)) . \tag{8}$$

With this definition, any differentiable equivariant map $\varphi : P\to F$ from a right K-manifold to a left K-manifold (i.e. any differentiable map $\varphi : P\to F$ satisfying the identity $\varphi(pk) = k^{-1}\varphi(p)$) induces a morphism $\varphi^*_K : A^*_K(F)\to A^*_K(P)$ of *differential* graded $\mathbb{R}$-algebras defined by pull-back of forms. In order to see this one uses the identity

$$\varphi_*(a^\#_p) = -a^\#_{\varphi(p)} \ \forall p \in P \ \forall a \in \mathfrak{k} \tag{9}$$

to show that for such a map φ one has $\varphi^* \circ \iota_{a^\#} = -\iota_{a^\#} \circ \varphi^*$ as maps from $A^*(F)$ to $A^*(P)$.

Let now $\pi : P \to B$ be a principal K-bundle on B. We have the standard $\mathbb{R}$-algebra isomorphism

$$H_K(\pi) : H^*(B, \mathbb{R}) \xrightarrow{\simeq} H^*_K(P, \mathbb{R}) \ .$$

Using the de Rham algebra of B and the Cartan algebra of the right K-manifold P, the isomorphism $H_K(\pi)$ corresponds to the map $H_K(\pi) : H^*_{\mathrm{DR}}(B) \to H^*_K(P)$ defined by

$$H_K(\pi)([\eta]_{\mathrm{DR}}) = [\pi^*(\eta)]_{\mathrm{C}} \ .$$

Here $\pi^*(\eta)$ is regarded as a constant, K-invariant $A^*(P)$-valued polynomial map on $\mathfrak{k}$, and $[\cdot]_{\mathrm{C}}$ denotes the Cartan cohomology class.

We will show that, choosing a connection A on the principal bundle P, one can give an explicit formula for the inverse map $H_K(\pi)^{-1} : H^*_K(P) \to H^*_{\mathrm{DR}}(B)$.

Fix a connection A on P. We recall that a k-form β on P is called horizontal if $\iota_X \beta = 0$ for any vertical tangent vector field X, and is called basic if it is horizontal and K-invariant. The algebra $A^*(P)_{\mathrm{ba}}$ of basic forms on P is d-invariant, and the differential graded algebra $(A^*(P)_{\mathrm{ba}}, d)$ can be identified with the differential graded algebra $(A^*(B), d)$ in the obvious way.

Denote by $(\cdot)^h$ the projection $T_P \to \Gamma_A$ of the tangent space of P on the horizontal distribution $\Gamma_A \subset T_P$ defined by A. We will use the same symbol for the linear operator which maps a differential form on P to the corresponding horizontal form, i.e.,

$$\beta^h(X_1, \dots, X_k) := \beta(X_1^h, \dots, X_k^h) \ .$$

Proposition 2.3. *Let A be a connection on P and $\Omega_A \in A^2(P, \mathfrak{k})$ its curvature form. For any $\alpha \in A^*_K(P)$ the form $\alpha(\Omega_A)^h \in A^*(P)$ is basic, the map $\alpha \mapsto \alpha(\Omega_A)^h$ defines a morphism of differential graded $\mathbb{R}$-algebras*

$$\pi^A_* : A^*_K(P) \to A^*(P)_{\mathrm{ba}} = A^*(B) \ ,$$

and the induced map $H(\pi^A_*) : H^*_K(P) \to H^*_{DR}(B)$ coincides with $H_K(\pi)^{-1}$.

This statement is similar to Proposition 7.34 in [BGV][2]. For completeness we include a short proof based on the same method.

Proof. The fact that $\alpha(\Omega_A)^h$ is basic follows from the invariance properties of α and Ω_A. The differential $d\big(\alpha(\Omega_A)^h\big)$ is automatically basic, so it suffices to compute it on horizontal vector fields, in other words it is sufficient to compute the covariant derivative $D_A\big(\alpha(\Omega_A)\big)$, where $D_A : A^*(P) \to A^*(P)$ is defined by $D_A(\sigma) = (d(\sigma^h))^h$. Fixing a basis $(a_i)_i$ in $\mathfrak{k}$, putting $\Omega_A = \Omega^i_A a_i$, and using Lemma 7.31 in [BGV] we obtain for a tensor product $f \otimes \beta \in \mathbb{R}[\mathfrak{k}] \otimes A^*(P)$

$$D_A\big((f \otimes \beta)(\Omega_A)\big) = D_A\big(f(\Omega_A) \wedge \beta\big) = \Big\{ d(f(\Omega_A) \wedge \beta) - \Omega^i_A \ \iota_{a_i^\#}(f(\Omega_A) \wedge \beta) \Big\}^h \ .$$

[2]Note however that, with conventions introduced in section 7.1 of [BGV], there is a sign error in Proposition 7.34 and its proof. The proof uses the same formula (8) for both left and right actions, and does not take into account the sign change (9).

Since $(d\Omega_A)^h = 0$ by Bianchi's identity and $\iota_{a_i^\#}\Omega_A = 0$ (because Ω_A is a horizontal form), we obtain

$$D_A\big((f\otimes\beta)(\Omega_A)\big) = \Big\{(f(\Omega_A)\wedge d\beta) - \Omega_A^i\ (f(\Omega_A)\wedge\iota_{a_i^\#}\beta)\Big\}^h$$
$$= \Big\{f(\Omega_A)\wedge(d\beta - \Omega_A^i\wedge\iota_{a_i^\#}\beta)\Big\}^h = \{d_K(f\otimes\beta)(\Omega_A)\}^h\ .$$

Therefore, for every $\alpha\in\mathbb{R}[\mathfrak{k}]\otimes A^*(P)$ (K-invariant or not) one has

$$D_A(\alpha(\Omega_A)) = \{(d_K(\alpha))(\Omega_A)\}^h\ .$$

This implies

$$(d\circ\pi_*^A)(\alpha) = (\pi_*^A\circ d_K)(\alpha)$$

for every $\alpha\in\big[\mathbb{R}[\mathfrak{k}]\otimes A^*(P)\big]^K$. The fact that the induced map $H(\pi_*^A): H_K^*(P)\to H_{\mathrm{DR}}^*(B)$ coincides with $H_K(\pi)^{-1}$ is now obvious, since for any form $\eta\in A^*(B)$, the basic form $\pi_*^A(\pi^*(\eta))$ coincides with $\pi^*(\eta)$, so the corresponding form in $A^*(B)$ is η. ∎

Remark 2.4. *The Chern-Weil morphism $\mathbb{R}[\mathfrak{k}]^K\to Z_{\mathrm{DR}}^*(B)$ can be obtained as the composition $\pi_*^A\circ j_K^P$, where $j_K^P:\mathbb{R}[\mathfrak{k}]^K\hookrightarrow A_K^*(P)$ is the obvious embedding. For an invariant polynomial $f\in\mathbb{R}[\mathfrak{k}]^K$, the cohomology class $[\pi_*^A(f)]_{\mathrm{DR}}$ is the characteristic class c_f associated with f.*

2.3. **Sections in Hamiltonian fibre bundles.** Let $\pi: P\to B$ be a principal K-bundle on B, F a manifold endowed with left K-action, and let $E := P\times_K F$ be the associated bundle with fibre F. In this section we study the morphism $H_K^*(F)\to H_{\mathrm{DR}}^*(B)$ defined by a section $\varphi\in\Gamma(B, P\times_K F)$.

It is well known that the data of a section φ in E is equivalent to the data of a K-equivariant map $P\to F$. We will denote the two objects by the same symbol. Using the formalism and the results of section 2.2, we obtain a morphism $H_K(\varphi_K^*): H_K^*(F)\to H_K^*(P)$, hence a morphism

$$H_K(\pi)^{-1}\circ H_K(\varphi_K^*): H_K^*(F)\to H_{\mathrm{DR}}^*(B)\ .$$

This morphism coincides with $H(\pi_*^A)\circ H_K(\varphi_K^*)$, and is given explicitly by the formula

$$[\alpha]_{\mathrm{C}}\mapsto\big[\{\varphi_K^*(\alpha)(\Omega_A)\}^h\big]$$

via the identification $A^*(B) = A^*(P)_{\mathrm{ba}}$. The following result was stated in an equivalent way in [LT2] section 6.2.2, where it was proved without using equivariant cohomology as a special case of the so-called the "generalized symplectic reduction".

Proposition 2.5. *Let ω be a K-invariant symplectic form on F, and $\mu: F\to\mathfrak{k}^\vee$ a moment map for the K-action on the symplectic manifold (F,ω). Denote by the same symbol also the associated co-moment map $\mathfrak{k}\to A^0(F)$, which is an invariant $A^0(F)$-valued linear map on $\mathfrak{k}$. Then:*

1. *$d_K(\omega-\mu) = 0$, hence $\omega-\mu$ defines a cohomology class $[\omega-\mu]_{\mathrm{C}}\in H_K^2(F)$.*
2. *The basic 2-form*
$$\{\varphi^*(\omega) - \langle\mu\circ\varphi, \Omega_A\rangle\}^h$$
is closed and represents the de Rham class
$$H_K(\pi)^{-1}\circ H_K(\varphi_K^*)([\omega-\mu]_{\mathrm{C}})\in H_{\mathrm{DR}}^2(B)\ .$$

Proof. One has $d_K(\omega-\mu)(a) = d((\omega-\mu)(a))+\iota_{a^\#}((\omega-\mu)(a)) = -d\mu^a+\iota_{a^\#}(\omega) = 0$. The second statement follows from Proposition 2.3. ∎

Suppose now that

$$\omega = d\theta \ , \tag{10}$$

where θ is a K-invariant 1-form on F [3]. In this case one has a natural moment map μ_θ for the K-action on (F,ω) defined by

$$\langle \mu_\theta, a\rangle := -\iota_{a^\#}\theta \ \ \forall a \in \mathfrak{k} \ . \tag{11}$$

Such a moment map is called exact (see [GGK] Example 2.8 p. 18). Note that θ defines an element of $A^1_K(F)$ and formulae (10), (11) are equivalent to

$$d_K(\theta) = \omega - \mu_\theta \ . \tag{12}$$

Therefore the cohomology class $[\omega-\mu_\theta]_{\mathrm{C}} \in H^*_K(F)$ hence also its image in $H^2_{\mathrm{DR}}(B)$ vanishes. More precisely we have

Corollary 2.6. *With the notations and assumptions of Proposition 2.5, suppose that $\omega = d\theta$, for a K-invariant 1-form on F, and let μ_θ be the corresponding exact moment map. Then $\varphi^*(\theta)^h$ is a basic 1-form on P and*

$$d(\varphi^*(\theta)^h) = \{\varphi^*(\omega) - \langle \mu_\theta\circ\varphi, \Omega_A\rangle\}^h \ .$$

Proof. Using (12) we obtain

$$d_K(\varphi^*\theta) = \varphi^*\omega - \mu_\theta\circ\varphi \ \text{ in } A^*_K(P) \ ,$$

hence $\pi^*_A(d_K(\varphi^*\theta)) = \pi^*_A(\varphi^*\omega - \mu_\theta\circ\varphi)$. The right hand term coincides with $\{\varphi^*(\omega) - \langle\mu_\theta\circ\varphi,\Omega_A\rangle\}^h$, and the left hand term coincides with $d(\varphi^*(\theta)^h)$ by Proposition 2.3. ∎

Suppose now that:

- B and F are complex manifolds and A is a (1,1)-connection on P, i.e., the curvature $F_A \in A^2(B,\mathrm{ad}(P))$ has type (1,1). This condition is equivalent to the integrability of the almost complex structure J_A on the complexification $Q := P\times_K K^{\mathbb{C}}$ (see [LT2] Appendix 7.1).
- The K-action on F is induced by a holomorphic action of the complex reductive group $K^{\mathbb{C}}$.

If these conditions are satisfied, we define:

Definition 2.7. *A section $\varphi : P\to F$ is called A-holomorphic if one of the following equivalent conditions is satisfied:*

- *$(d\varphi)^h : \Gamma_A \to T_F$ commutes with the almost complex structure induced via $\pi_* : \Gamma_A\to T_B$ on the horizontal distribution Γ_A of A.*
- *the $K^{\mathbb{C}}$-equivariant map $Q\to F$ induced by φ is J_A-holomorphic.*

[3] This happens for instance when (F,ω) is a Hermitian vector space endowed with the canonical Kählerian form, or when F is the cotangent bundle of a K-manifold endowed with the canonical symplectic structure.

Suppose now that $\omega = \frac{1}{4} dd^c \nu$, where ν is a K-invariant real function on F, and the operator d^c is defined as $d^c := i(\bar\partial - \partial) = J^{-1} dJ$. For a tangent vector X and a smooth function σ one has $d^c(\sigma)(X) = -d\sigma(JX)$.

In this case we obtain an obvious integral of ω – namely $\theta_\nu := \frac{1}{4} d^c \nu$ – hence also an associated exact moment map μ_{θ_ν} which we simply denote by μ_ν.

For an equivariant map $\varphi : P \to F$ we denote by $\nu(\varphi) \in A^0(B)$ the real function defined by $\nu(\varphi) \circ \pi = \nu \circ \varphi$. If now $\varphi : P \to F$ is A-holomorphic, we obtain

$$\{\varphi^*(d^c \nu)\}^h = \pi^*(d^c(\nu(\varphi))) \tag{13}$$

by evaluating both sides on a horizontal tangent vector.

Corollary 2.8. *With the notations and under the assumptions above suppose that $\omega = \frac{1}{4} dd^c \nu$, A is a (1,1)-connection on P, and φ is A-holomorphic. Then*

$$\frac{1}{4} dd^c(\nu(\varphi)) = \{\varphi^*(\omega) - \langle \mu_\nu \circ \varphi, \Omega_A \rangle\}^h \ , \tag{14}$$

where the basic form on the right has been regarded as a 2-form on B.

Example: Let (F, h_F) be a Hermitian vector space, and $\nu_F : F \to \mathbb{R}$ the $\mathrm{U}(F)$-invariant real function defined by $\nu_F(v) := h_F(v, v)$. The Kähler form ω_F associated with h_F can be written as

$$\omega_F = \frac{i}{2} \partial\bar\partial \nu_F = \frac{1}{4} dd^c \nu_F \ .$$

Put $\theta_F := \frac{1}{4} d^c \nu_F$. Note that θ_F is $\mathrm{U}(F)$-invariant and

$$\omega_F = d\theta_F \ . \tag{15}$$

Let now $\rho : K \to \mathrm{U}(F)$ be a unitary representation, and let $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ be an ad-invariant inner product on the Lie algebra $\mathfrak{k}$ of K. *The standard moment map* $\mu_0 : F \to \mathfrak{k}^\vee$ for the K-action on F defined by ρ is the exact moment map associated with the K-invariant form θ_F, i.e., it is given by

$$\langle \mu_0, a \rangle = -\iota_{a^\#} \theta_F \ \forall a \in \mathfrak{k} \ .$$

A simple computation shows that

$$\mu_0(v)(a) = \left\langle \rho^* \left(-\frac{i}{2} v \otimes v^* \right), a \right\rangle_{\mathfrak{k}} = \left\langle -\frac{i}{2} v \otimes v^*, \rho_*(a) \right\rangle_{\mathrm{u}(F)} \ ,$$

where the adjoint is computed with respect to $\langle \cdot, \cdot \rangle_{\mathfrak{k}}$ and the standard inner product $\langle \cdot, \cdot \rangle_{\mathrm{u}(F)}$ on the Lie algebra $\mathrm{u}(F)$ of skew-Hermitian endomorphisms of F.

Definition 2.9. *A unitary representation $\rho : K \to U(F)$ is said to contain the homotheties of F if there exists a K-invariant element $a_0 \in \mathfrak{k}$ such that $\rho_*(a_0) = i\mathrm{id}_F$.*

For such an element a_0 one has

$$\mu_0(v)(-2a_0) = \langle v \otimes v^*, \mathrm{id}_F \rangle_{i\mathrm{u}(F)} = h_F(v, v) = \nu_F(v) \ . \tag{16}$$

2.4. **Families of connections and sections.** Let $\mathcal{C}$ be a manifold and $\mathcal{G}$ a Lie group with a free action on $\mathcal{C}$ such that the quotient manifold $\mathcal{B} := \mathcal{C}/\mathcal{G}$ exists and the projection $\mathcal{C} \to \mathcal{B}$ becomes a principal $\mathcal{G}$-bundle. We allow both $\mathcal{C}$ and $\mathcal{G}$ to be infinite dimensional. Fix a connection Γ_0 in this principal bundle.

Let X be a closed d-dimensional manifold, P a principal K-bundle on X, and

$$u : \mathcal{G} \to \mathrm{Aut}(P) = \Gamma(X, P \times_{\mathrm{Ad}} K)$$

a morphism of Lie groups. It is important that u is not supposed to be surjective; in many interesting examples $\mathcal{G}$ is the subgroup of $\mathrm{Aut}(P)$ associated with a normal subgroup of K. Denote by $\mathcal{A}(P)$ the space of connections on P.

Definition 2.10. *A smooth $\mathcal{G}$-equivariant family of connections on P is a smooth map $\mathfrak{a} : \mathcal{C} \to \mathcal{A}(P)$ given by a family $(A_c)_{c\in\mathcal{C}}$ such that*

$$A_{gc} = u(g)_*(A_c) \ \forall g \in \mathcal{G} \ \forall c \in \mathcal{C} \ . \tag{17}$$

Examples: 1. Let E, E_0 be Hermitian vector bundles of rank r, r_0 on X and A_0 a fixed connection on A_0. A pair of type (E, E_0) is a pair (A, φ) consisting of a Hermitian connection on E and a section $\varphi \in A^0(X, E^\vee \otimes E_0)$. In [OT2] we gave a gauge theoretical construction of certain Quot spaces. In this construction we used as configuration space the space of irreducible pairs of type (E, E_0), i.e., the subspace of pairs $(A, \varphi) \in \mathcal{A}(E) \times A^0(X, E^\vee \otimes E_0)$ with trivial stabilizer with respect to the gauge group $\mathcal{G} := \Gamma(X, \mathrm{U}(E))$.

In this case we have $K = \mathrm{U}(r) \times \mathrm{U}(r_0)$, $P := P_E \times_X P_{E_0}$ is the product of the frame bundles associated with E and E_0, and u is the obvious embedding of $\mathcal{G}$ in $\mathrm{Aut}(P)$. The map $\mathfrak{a}$ sends a pair (A, φ) to the product connection $A \times A_0$. The configuration space and the gauge group of the classical vortex equation are obtained as special case of this construction taking (E_0, A_0) to be the trivial Hermitian line bundle endowed with the trivial connection.

2. With the same notations as above let now a_0 be a fixed connection on the determinant $\mathrm{U}(1)$-bundle $\det(E)$. We denote by $\mathcal{A}_{a_0}(E)$ the space of a_0-oriented connections on E, i.e., the space of Hermitian connections on E inducing a_0 on $\det(E)$. In the theory of master spaces [OT1] one considers as configuration space the space of *irreducible a_0-oriented pairs* of type (E, E_0), i.e., the subspace of pairs $(A, \varphi) \in \mathcal{A}_{a_0}(E) \times \Gamma(X, E^\vee \otimes E_0))$ with trivial stabilizer with respect to the natural action of the gauge group $\mathcal{G} := \Gamma(X, \mathrm{SU}(E))$.

Put $\mathfrak{P} := \mathcal{C} \times P$ and note that the projection $\pi : \mathfrak{P} \to \mathcal{C} \times X$ is a principal K-bundle. We define left $\mathcal{G}$-actions on $\mathcal{C} \times X$ and $\mathfrak{P}$ by

$$g(c, x) := (gc, x) \ , \ g(c, p) := (gc, u(g)(p)) \ .$$

The bundle $\mathfrak{P}$ comes with a tautological connection $\mathfrak{A}$ whose horizontal space at a point $(c, p) \in \mathfrak{P}$ is $T_c\mathcal{C} \times \Gamma_p^{A_c}$. In other words $\mathfrak{A}$ agrees with the trivial (product) connection in the $\mathcal{C}$ direction (in which $\mathfrak{P}$ is trivial), and for every $c \in \mathcal{C}$ its restriction to $\{c\} \times P$ coincides with A_c. Consider points $c \in \mathcal{C}$, $x \in X$, $p \in P_x$ and tangent vectors α_1, $\alpha_2 \in T_c\mathcal{C}$, v_1, $v_2 \in T_xX$. Denote by $\tilde{v}_1$, $\tilde{v}_2 \in T_pP$ the A_c-horizontal lifts of v_1, v_2. Then one has

$$\Omega_{\mathfrak{A}}((\alpha_1, \tilde{v}_1), (\alpha_2, \tilde{v}_2)) = \Omega_{A_c}(\tilde{v}_1, \tilde{v}_2) + \iota_{v_2}\mathfrak{a}_*(\alpha_1) - \iota_{v_1}\mathfrak{a}_*(\alpha_2) \ . \tag{18}$$

The equivariance property (17) implies that $\mathfrak{A}$ is $\mathcal{G}$-invariant. On the other hand, the fixed connection Γ_0 on the principal $\mathcal{G}$-bundle $\mathcal{C} \to \mathcal{B}$ defines a connection Γ on the principal $\mathcal{G}$-bundle $\mathcal{C} \times X \to \mathcal{B} \times X$, whose horizontal distribution is given by $\Gamma_{c,x} := \Gamma_{0,c} \oplus T_x X$. The curvature of Γ is

$$\Omega_\Gamma((\alpha_1, v_1), (\alpha_2, v_2)) = \Omega_{\Gamma_0}(\alpha_1, \alpha_2) \ \ \forall \alpha_1,\ \alpha_2 \in \Gamma_{0,c},\ \forall v_1,\ v_2 \in T_x X \ . \tag{19}$$

Denote by $\mathbb{P}$ the quotient bundle $\mathbb{P} := \bar{\mathfrak{P}}$ on the quotient manifold $\mathcal{B} \times X$. As explained in section 2.1, the connection Γ can be used to define the quotient connection $\mathbb{A} = \bar{\mathfrak{A}}$ on the quotient K-bundle $\bar{\pi} : \mathbb{P} \to \mathcal{B} \times X$, whose curvature is given by Proposition 2.2. In order to write down explicitly this curvature, we need an explicit formula for the second term of the right hand side in (6).

Remark 2.11. *Let $\nu \in \mathrm{Lie}(\mathcal{G})$, $u_*(\nu) \in A^0(P \times_{\mathrm{ad}} \mathfrak{k})$ be the corresponding infinitesimal automorphism of P, $\nu^\#_{\mathfrak{P}}$ the induced vector field on $\mathfrak{P} = \mathcal{C} \times P$, and let $\mathfrak{p} = (c, p) \in \mathfrak{P}$. Then*

$$\theta_{\mathfrak{A}}(\nu^\#_{\mathfrak{P},\mathfrak{p}}) = u_*(\nu)_p \ ,$$

where on the right $u_*(\nu)$ has been regarded as a K-equivariant map $P \to \mathfrak{k}$ and $u_*(\nu)_p \in \mathfrak{k}$ is the value of this map at p.

Proof. Indeed, $\nu^\#_{\mathfrak{P},\mathfrak{p}} = (\nu^\#_{\mathcal{C},c}, u_*(\nu)^\#_p) = (\nu^\#_{\mathcal{C},c}, (u_*(\nu)_p)^\#_p)$. Since $(\nu^\#_{\mathcal{C},c}, 0)$ is $\mathfrak{A}$-horizontal and $\theta_{\mathfrak{A}}$ agrees with the connection form θ_{A_c} on $\{c\} \times P$, the claim follows from the properties of a connection form. ∎

Applying Proposition 2.2 to our situation we get

Proposition 2.12. *Let $c \in \mathcal{C}$, $[c]$ its image in $\mathcal{B}$. Let a_1, $a_2 \in T_{[c]}\mathcal{B}$, and denote by α_i the corresponding Γ_0-horizontal lifts. Consider points $x \in X$, $p \in P_x$, tangent vectors v_1, $v_2 \in T_x X$, and let $\tilde{v}_i \in \Gamma_p^{A_c} \in T_p P$ be their A_c-horizontal lifts. Put $\xi_i := (a_i, v_i) \in T_{([c],x)}(\mathcal{B} \times X)$ and denote by $\tilde{\xi}_i \in T_{[c,p]}\mathbb{P}$ their $\mathbb{A}$-horizontal lifts. Then*

$$\Omega_{\mathbb{A}}(\tilde{\xi}_1, \tilde{\xi}_2) = \Omega_{A_c}(\tilde{v}_1, \tilde{v}_2) + \iota_{v_2}\mathfrak{a}_*(\alpha_1) - \iota_{v_1}\mathfrak{a}_*(\alpha_2) + \{u_*\Omega_{\Gamma_0}(\alpha_1, \alpha_2)\}_p \ . \tag{20}$$

Proof. Indeed, Proposition 2.2 together with Remark 2.11 gives

$$\begin{aligned}\Omega_{\mathbb{A}}(\tilde{\xi}_1, \tilde{\xi}_2) &= \Omega_{\mathfrak{A}}((\alpha_1, \tilde{v}_1), (\alpha_2, \tilde{v}_2)) + \theta_{\mathfrak{A}}(\Omega_\Gamma((\alpha_1, v_1), (\alpha_2, v_2))^\#_{\mathfrak{P},(c,p)}) \\ &= \Omega_{\mathfrak{A}}((\alpha_1, \tilde{v}_1), (\alpha_2, \tilde{v}_2)) + \{u_*\Omega_\Gamma((\alpha_1, v_1), (\alpha_2, v_2))\}_p \ .\end{aligned}$$

Now use (18) and (19). ∎

As a corollary we obtain the following formula which computes characteristic forms of degree 2 of the connection $\mathbb{A}$.

Corollary 2.13. *Let $h : \mathfrak{k} \times \mathfrak{k} \to \mathbb{R}$ be an ad-invariant symmetric bilinear form. Let $c \in \mathcal{C}$, $x \in X$, $p \in P_x$, a_1, $a_2 \in T_{[c]}\mathcal{B}$, v_1, $v_2 \in T_x X$, and let $\alpha_i \in \Gamma_{0,c}$, $\tilde{v}_i \in \Gamma_p^{A_c}$ be horizontal lifts of a_i, v_i with respect to the connections Γ_0, A_c respectively. Denote by $h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}}) \in A^4(\mathcal{B} \times X)$ the characteristic form obtained from $\Omega_{\mathbb{A}}$ by coupling the wedge square with h. Then:*

$$\begin{aligned}-\frac{1}{2}h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}})(a_1, a_2, v_1, v_2) = h(\mathfrak{a}_*(\alpha_1) \wedge \mathfrak{a}_*(\alpha_2))(v_1, v_2) - \\ - h(u_*\Omega_{\Gamma_0}(\alpha_1, \alpha_2)_p, \Omega_{A_c})(\tilde{v}_1, \tilde{v}_2) \ .\end{aligned}$$

Proof. See [DK, section 5.2.3], [LT2, section 6.2.2]. ■

Note that the term $h(u_*\Omega_{\Gamma_0}(\alpha_1,\alpha_2)_p,\Omega_{A_c})(\tilde v_1,\tilde v_2)$ is independent of the choice of p in the fibre P_x, so we can omit the symbol p in this expression and write $h(u_*(\Omega_\Gamma(\alpha_1,\alpha_2)),\Omega_{A_c})(v_1,v_2)$. With this convention the formula above yields the following identity in $A^2(X)$:

$$-\frac{1}{2}h(\Omega_{\mathbb{A}}\wedge\Omega_{\mathbb{A}})(a_1,a_2)\Big|_{\{[c]\}\times X}=h(\mathfrak{a}_*(\alpha_1)\wedge\mathfrak{a}_*(\alpha_2))-h(u_*\Omega_{\Gamma_0}(\alpha_1,\alpha_2),\Omega_{A_c})\ . \tag{21}$$

Let now be F be a differentiable manifold endowed with a left K-action and let $E:=P\times_K F$ be the associated bundle with fibre F. Let $(\varphi_c)_{c\in\mathcal{C}}$ be a *$\mathcal{G}$-equivariant family of sections in E parameterized by our configuration space* $\mathcal{C}$, i.e., a smooth map $\phi:\mathcal{C}\to\Gamma(X,E)$ such that

$$\varphi_{gc}=u(g)_*(\varphi_c)\ \ \forall g\in\mathcal{G},\ \forall c\in\mathcal{C}\ .$$

Here we denote by $u(g)_*:\Gamma(X,E)\to\Gamma(X,E)$ the push-forward map associated with g. Identifying sections in E with the corresponding K-equivariant maps $P\to F$, we can write

$$u(g)_*(\varphi)=\varphi\circ g^{-1}\ .$$

Note that such a family ϕ defines a $\mathcal{G}$-invariant, K-equivariant map $\mathcal{C}\times P\to F$ given by $(c,p)\mapsto\varphi_c(p)$. This map descends to a K-equivariant map $\mathbb{P}=\mathcal{C}\times P/\mathcal{G}\to F$, i.e., to a section Φ in the universal F-bundle $\mathbb{E}:=\mathbb{P}\times_K F$ over $\mathcal{B}\times X$, which will be called *the universal section of the family.*

Suppose now that ω_F is a symplectic form on F, the K-action on F is symplectic and admits a moment map $\mu_F:F\to\mathfrak{k}^\vee$.

By Proposition 2.5 applied to the universal section Φ and the quotient connection $\mathbb{A}$ on the principal K-bundle $\bar\pi:\mathbb{P}\to\mathcal{B}\times X$ it follows that the basic form

$$\{\Phi^*(\omega_F)-\langle\mu_F\circ\Phi,\Omega_{\mathbb{A}}\rangle\}^h$$

is closed and represents the cohomology class

$$H_K(\bar\pi)^{-1}\circ H_K(\Phi)([\omega_F-\mu_F]_{\mathrm{C}})\in H^2_{\mathrm{DR}}(\mathcal{B}\times X)\ .$$

Let $\eta=\eta(\phi,\omega_F,\mu_F)\in Z^2_{\mathrm{D}R}(\mathcal{B}\times X)$ be the *closed* form on $\mathcal{B}\times X$ which corresponds to this basic form. For tangent vectors $a_i\in T_{[c]}\mathcal{B}$, horizontal lifts $\alpha_i\in\Gamma_{0,c}$ of a_i, and $p\in P$ we have

$$\eta_{([c],x)}(a_1,a_2)=\omega_F(\Phi_*(\alpha_1),\Phi_*(\alpha_2))-\langle\mu_F(\varphi_c(p)),u_*\Omega_{\Gamma_0}(\alpha_1,\alpha_2)_p\rangle\ . \tag{22}$$

For a fixed point $x\in X$ the term $\langle\mu(\varphi_c(p)),u_*\Omega_{\Gamma_0}(\alpha_1,\alpha_2)_p\rangle$ is independent of the choice of $p\in P_x$, so we can omit the symbol p and interpret the expression $\langle\mu(\varphi_c),u_*\Omega_{\Gamma_0}(\alpha_1,\alpha_2)\rangle$ as a smooth function on X. With this convention, formula (22) becomes

$$\eta(a_1,a_2)\big|_{\{[c]\}\times X}=\omega_F(\Phi_*(\alpha_1),\Phi_*(\alpha_2))-\langle\mu_F(\varphi_c),u_*\Omega_{\Gamma_0}(\alpha_1,\alpha_2)\rangle\ . \tag{23}$$

Now we fix forms $w \in A^{d-2}(X)$, $v \in A^d(X)$, and a K-invariant symmetric bilinear form h on $\mathfrak{k}$. Define $\mathcal{G}$-invariant closed 2-forms ω_v, $\omega_w \in A^2(\mathcal{C})$ by

$$\omega_w(\alpha_1,\alpha_2) := \int_X h(\mathfrak{a}_*(\alpha_1),\mathfrak{a}_*(\alpha_2)) \wedge w \ , \ \omega_v(\alpha_1,\alpha_2) := \int_X \omega_F(\Phi_*(\alpha_1),\Phi_*(\alpha_2)) v \ .$$

The horizontal projections ω_w^h, ω_v^h with respect to the connection Γ are basic forms on the $\mathcal{G}$-bundle $\mathcal{C} \to \mathcal{B}$, so they can be interpreted as 2-forms on the base $\mathcal{B}$ of this bundle. Note that in general these forms are not closed.

We define $\mu_{v,w} : \mathcal{C} \to \mathrm{Lie}(\mathcal{G})^\vee$ by

$$\langle \mu_{v,w}(c), s\rangle := \int_X h(u_*(s), \Omega_{A_c} \wedge w) + \langle \mu_F(\varphi_c), u_*(s)\rangle v \ .$$

When w is closed, this map satisfies the axioms of a moment map for the $\mathcal{G}$-action on $\mathcal{C}$ with respect to the closed form $\omega_{v,w} := \omega_w + \omega_v$ which of course can be degenerate in general.

Denote by $p_{\mathcal{B}*} : A^*(\mathcal{B} \times X) \to A^*(\mathcal{B})$ fibre integration associated with the projection $p_{\mathcal{B}} : \mathcal{B} \times X \to \mathcal{B}$. Using formulae (21), (23) and the definition of $\mu_{v,w}$ we obtain the following important identity:

$$\omega_{v,w}^h = p_{\mathcal{B}*}\left[-\frac{1}{2}[h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}})] \wedge p_X^*(w) + \eta \wedge p_X^*(v)\right] + p_{\mathcal{B}^*}\langle \mu_{v,w}, \Omega_{\Gamma_0}\rangle \ . \tag{24}$$

This proves:

Theorem 2.14. *The restriction of the form $\omega_{v,w}^h$ to the quotient $\mathcal{N} := Z(\mu_{v,w})/\mathcal{G}$ is closed and is given by*

$$\omega_{v,w}^h\big|_{\mathcal{N}} = p_{\mathcal{N}*}\left[-\frac{1}{2}[h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}})] \wedge p_X^*(w) + \eta \wedge p_X^*(v)\right] \ .$$

By Corollary 2.6 we know that the form η is exact when the moment map μ_F is exact. Hence:

Corollary 2.15. *Suppose w is closed, $\mu_F = \mu_0 + \tau$, where μ_0 is exact and $\tau \in \mathfrak{k}^\vee$ is K-invariant. Then one has*

$$[\omega_{v,w}^h\big|_{\mathcal{N}}] = p_{\mathcal{N}*}\left[-\frac{1}{2}[h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}}] \cup p_X^*[w] - [\tau(\Omega_{\mathbb{A}})] \cup p_X^*[v]\right]$$

in $H^2_{\mathrm{DR}}(\mathcal{N})$.

Note that each term of the right hand side of this formula can be written as the slant product of a characteristic class of the bundle $\mathbb{P}$ with a homology class of X.

3. Applications

3.1. **Kähler forms of canonical Hermitian metrics and tautological classes.**

3.1.1. *The general setting.* We now apply the results obtained in section 2.4 to the families of connections and sections associated with the universal gauge theoretical problem introduced in section 1. We will obtain the main theorems Theorem 3.1 and Theorem 3.2 stated in the introduction as corollaries of Theorem 2.14 above.

Let $r : K \to K_0$ be an epimorphism of compact Lie groups with kernel N. Fix a K-invariant inner product k on $\mathfrak{k}$ and denote by $p_{\mathfrak{n}} : \mathfrak{k} \to \mathfrak{n}$ the k-orthogonal projection onto the Lie algebra $\mathfrak{n}$ of N. The symmetric bilinear form $h:\mathfrak{k}\times\mathfrak{k}\to\mathbb{R}$ defined by

$$h(x,y):=k(p_{\mathfrak{n}}(x),p_{\mathfrak{n}}(y))$$

is K-invariant and non-degenerate on $\mathfrak{n}$. Suppose we are given a left K-action on a Kähler manifold (F,J_F,g_F) by holomorphic isometries, and a moment map μ_F for this action on the corresponding symplectic manifold (F,ω_F). We denote by $m_F:F\to\mathfrak{k}$ the $\mathfrak{k}$-valued map defined by $\mu_F=k(m_F,\cdot)$.

Let now (X,J,g) be an n-dimensional Kähler manifold, $\pi:P\to X$ a principal K-bundle on X, and $P_0:=P\times_K K_0$ the associated K_0-bundle. We fix a connection $A_0\in\mathcal{A}(P_0)$ of type $(1,1)$ (our "orientation data").

Let $E:=P\times_K F$ be the associated bundle with fibre F. The gauge group of our moduli problem is

$$\mathcal{G}:=\Gamma(X,P\times_K N)$$

and acts from the left on the space of A_0-oriented pairs $\mathcal{A}_{A_0}(P)\times\Gamma(X,E)$. This space has a natural Kähler metric depending on the triple (k,g,g_F) whose Kähler form is given by the formula ([LT2] p. 66):

$$\Omega((\alpha,\psi),(\beta,\chi):=\int_X h(\alpha\wedge\beta)\wedge\omega_g^{n-1}+(n-1)!\int_X\omega_F(\psi,\chi)\mathrm{vol}_g\ .$$

In order to avoid technical problems we will concentrate on the configuration space of irreducible oriented pairs. Let $\mathcal{C}^*:=[\mathcal{A}_{A_0}(P)\times\Gamma(X,E)]^*$ be the open subspace of irreducible oriented pairs , i.e., of pairs $(A,\varphi)\in\mathcal{A}_{A_0}(P)\times\Gamma(X,E)$ with trivial stabilizer with respect to the $\mathcal{G}$-action. After appropriate Sobolev completions the projection $\mathcal{C}^*\to\mathcal{B}^*:=\mathcal{C}^*/\mathcal{G}$ becomes a principal $\mathcal{G}$-bundle. We refer to [LT1, section 4.2] for details. We endow this bundle with the connection Γ_0 defined by L^2-orthogonality to the $\mathcal{G}$-orbits[4] [LT2].

Note that we have a tautological family $(A_c)_{c\in\mathcal{C}^*}$ of connections, and a tautological family $(\varphi_c)_{c\in\mathcal{C}^*}$ of sections parameterized by $\mathcal{C}^*$. The corresponding maps $\mathfrak{a}$, ϕ are given by the projections onto the two factors.

The gauge theoretical moduli space $\mathcal{M}^*$ corresponding to the data above is the space of equivalence classes of pairs $(A,\varphi)\in\mathcal{C}^*$ satisfying the generalized vortex equations

$$\left\{\begin{array}{ccc} F_A^{02} & = & 0\\ \varphi \text{ is } A\text{-holomorphic} & & \\ p_{\mathfrak{n}}\left[\Lambda_g F_A+m_F(\varphi)\right] & = & 0\ . \end{array}\right. \qquad (V)$$

In the third equation we used the same symbol for the bundle epimorphism $P\times_K\mathfrak{k}\to P\times_K\mathfrak{n}$ induced by $p_{\mathfrak{n}}:\mathfrak{k}\to\mathfrak{n}$.

Taking $w=\omega_g^{n-1}$ and $v=(n-1)!\mathrm{vol}_g=\frac{1}{n}\omega_g^n$, we obtain for $s\in\mathrm{Lie}(\mathcal{G})=A^0(P\times_K\mathfrak{n})$

$$\begin{aligned}\langle\mu_{v,w}(A,\varphi)),s\rangle&=\int_X h(s,F_A\wedge\omega_g^{n-1})+(n-1)!\langle\mu_F(\varphi),s\rangle\mathrm{vol}_g\\&=\int_X h(s,F_A\wedge\omega_g^{n-1})+(n-1)!k(s,m_F(\varphi))\mathrm{vol}_g\end{aligned}$$

[4]The construction can be extended to the non-Kählerian framework and yields a strongly KT Hermitian metric on the moduli space, but one has to use a different connection Γ_0 [LT1], [LT2].

$$= (n-1)! \int_X \{h(s, p_{\mathfrak{n}}\Lambda_g F_A) + k(s, p_{\mathfrak{n}}(m_F(\varphi))\} \operatorname{vol}_g$$
$$= (n-1)! \int_X \{h(s, p_{\mathfrak{n}}\Lambda_g F_A) + h(s, p_{\mathfrak{n}}(m_F(\varphi))\} \operatorname{vol}_g$$
$$= (n-1)! \int_X h(s, p_{\mathfrak{n}}(\Lambda_g F_A + m_F(\varphi)))\operatorname{vol}_g.$$

This shows that the third equation in (V) is equivalent to the vanishing of the moment map $\mu_{v,w}$ introduced above. On the other hand, using the fact that the restriction of h to $\mathfrak{n}\times\mathfrak{n}$ is an inner product one can prove easily that the restriction of the form $\omega^h_{v,w}$ on $\mathcal{M}^*$ is non-degenerate. This form coincides with the Kähler form of the standard Hermitian metric $g_{\mathcal{M}^*}$ on $\mathcal{M}^*$ associated with the pair $(h|_{\mathfrak{n}\times\mathfrak{n}}, g_F)$ (see [LT2, section 6.2.2]). Therefore, using Theorem 2.14 and Corollary 2.15 we obtain

Theorem 3.1. *The canonical Hermitian metric on $\mathcal{M}^*$ is Kähler and its Kähler form is*

$$\omega_{\mathcal{M}^*} = p_{\mathcal{M}^*,*}\left[-\frac{1}{2}[h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}})] \wedge p_X^*(\omega_g^{n-1}) + (n-1)!\eta \wedge p_X^*(\operatorname{vol}_g)\right] ,$$

with $\eta := \{\Phi^*(\omega_F) - \langle \mu_F \circ \Phi, \Omega_{\mathbb{A}}\rangle\}^h$, which is a closed 2-form. If μ_F can be written as $\mu_F = \mu_0 + \tau$, where μ_0 is an exact moment map and $\tau \in \mathfrak{k}^\vee$ is K-invariant, then the de Rham cohomology class $[\omega_{\mathcal{M}^*}]_{\mathrm{DR}}$ is given by

$$[\omega_{\mathcal{M}^*}]_{\mathrm{DR}} = p_{\mathcal{M}^*,*}\left[-\frac{1}{2}[h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}})] \cup p_X^*[\omega_g]^{n-1} - (n-1)![\tau(\Omega_{\mathbb{A}})] \cup p_X^*[\operatorname{vol}_g]\right] .$$

Note that, by formula (23) the term $\tau(\Omega_{\mathbb{A}})$ coincides with $\tau(\Omega_{\Gamma_0})$, hence this term depends only on the restriction $\tau_{|\mathfrak{n}}$.

Theorem 3.2. *Let F be a Hermitian vector space, and $\rho : K \to \mathrm{U}(F)$ a unitary representation. For a K-invariant linear form $\tau \in \mathfrak{k}^\vee$ put $\mu_F := \mu_0 + \tau$, where μ_0 is the standard moment map for the K-action on F.*

Suppose there exists a K-invariant element $a_0 \in \mathfrak{n}$ such that $\rho_*(a_0) = i\mathrm{id}_F$. Then the Kähler form $\omega_{\mathcal{M}^*}$ is given by

$$\omega_{\mathcal{M}^*} = p_{\mathcal{M}^*,*}\left[-\frac{1}{2}h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}}) \wedge p_X^*(\omega_g)^{n-1} - (n-1)!\tau(\Omega_{\mathbb{A}}) \wedge p_X^*(\operatorname{vol}_g)\right] ,$$

hence it coincides with the fibre integration of a characteristic form of the connection $\mathbb{A}$ on the universal bundle $\mathbb{P}$.

Proof. Since $\mu_F = \mu_0 + \tau$ and the form η depends linearly on μ_F one has

$$\omega_{\mathcal{M}^*} - p_{\mathcal{M}^*,*}\left[-\frac{1}{2}h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}}) \wedge p_X^*(\omega_g)^{n-1} - (n-1)!\tau(\Omega_{\mathbb{A}}) \wedge p_X^*(\operatorname{vol}_g)\right]$$
$$= (n-1)! p_{\mathcal{M}^*,*}\left(\eta_0 \wedge p_X^*(\operatorname{vol}_g)\right) ,$$

where

$$\eta_0 = \{\Phi^*(\omega_F) - \langle \mu_0 \circ \Phi, \Omega_{\mathbb{A}}\rangle\}^h .$$

By Corollary 2.8 we obtain on $\mathcal{B}^* \times X$ the identity

$$\eta_0 = \frac{1}{4} dd^c |\Phi|^2 ,$$

where the universal section Φ has been regarded as a section in the Hermitian vector bundle $\mathbb{E} := \mathbb{P} \times_K F$ on $\mathcal{B}^* \times X$. Therefore it suffices to prove that

$$p_{\mathcal{M}^*,*}(dd^c|\Phi|^2 \wedge p_X^*(\mathrm{vol}_g)) = 0 \ .$$

Since the operators d, d^c commute with the fibre integration associated with a *holomorphic* map we have

$$p_{\mathcal{M}^*,*}(dd^c|\Phi|^2 \wedge p_X^*(\mathrm{vol}_g)) = dd^c \left\{p_{\mathcal{M}^*,*}(|\Phi|^2 \wedge p_X^*(\mathrm{vol}_g))\right\} \ .$$

Since $a_0 \in \mathfrak{n}$ is K-invariant by assumption, it defines a section $\tilde{a}_0 \in A^0(P \times_K \mathfrak{n})$. Taking the L^2 inner product of the third equation in (V) with $\tilde{a}_0$ we compute

$$0 = \langle \Lambda F_A, \tilde{a}_0\rangle_{L^2} + \langle m_0(\varphi), \tilde{a}_0\rangle_{L^2} + \tau(a_0)\mathrm{Vol}_g \ .$$

The first term is a characteristic number of P, so is a constant. On the other hand by formula (16) we obtain:

$$\langle m_0(\varphi), \tilde{a}_0\rangle_{L^2} = -\frac{1}{2}\int_X |\varphi|^2 \mathrm{vol}_g \ .$$

This implies that the map

$$[A, \varphi] \mapsto \int_X |\varphi|^2$$

is constant on $\mathcal{M}$. But this map coincides with $p_{\mathcal{M}^*,*}(|\Phi|^2 \wedge p_X^*(\mathrm{vol}_g))$. ■

3.1.2. *Vortex moduli spaces.* Let E, E_0 be Hermitian vector bundles on X of ranks r, r_0, P_E, P_{E_0} the associated frame bundles, and $t \in \mathbb{R}$. Fix a Hermitian connection A_0 of type $(1,1)$ on E_0, and let $\mathcal{M}_t = \mathcal{M}_t(E, E_0, A_0)$ be the moduli space of pairs $(B, s) \in \mathcal{A}(E) \times A^0(\mathrm{Hom}(E, E_0))$ solving the t-vortex equation:

$$\left\{\begin{array}{ccc} F_B^{02} & = & 0 \\ \bar{\partial}_{B,A_0} s & = & 0 \\ i\Lambda F_A - \frac{1}{2}s^* \circ s + t\mathrm{id}_E & = & 0 \end{array}\right. \qquad (V_t(E, E_0, A_0))$$

modulo the gauge group $\mathcal{G} := \Gamma(X, \mathrm{U}(E))$. This moduli problem can be obtained as a special case of the universal moduli problem (V) considered in Theorem 3.2 by taking $K = \mathrm{U}(r) \times \mathrm{U}(r_0)$, $N = \mathrm{U}(r)$, $F := \mathrm{Hom}(\mathbb{C}^r, \mathbb{C}^{r_0})$ with the obvious unitary representation $\rho : K \to \mathrm{U}(F)$, with bundle $P := P_E \times_X P_{E_0}$, and moment map

$$m_F(f) = (\frac{i}{2}f^* \circ f - it\mathrm{id}_{\mathbb{C}^r}, -\frac{i}{2}f \circ f^*) \ .$$

Note that the space $\mathcal{A}_{A_0}(P)$ can be identified with the space $\mathcal{A}(E)$ of Hermitian connections on E. We put $\mathcal{C}^* := [\mathcal{A}(E) \times A^0(\mathrm{Hom}(E, E_0))]^*$ and use the quotient construction to define a universal connection $\mathbb{B}$ on the universal vector bundle

$$\mathbb{E} := {}^{\mathcal{C}^* \times E}\!/_{\mathcal{G}}$$

over $\mathcal{B}^* \times X$.

We use the standard inner product $k((a, a_0), (b, b_0)) = -\mathrm{Tr}(ab) - \mathrm{Tr}(a_0 b_0)$ on $\mathrm{u}(r) \oplus \mathrm{u}(r_0)$. With this choice we get $h((a, a_0), (b, b_0)) = -\mathrm{Tr}(ab)$ and $\mu_F = \mu_0 + \tau$, where $\tau(a, a_0) = it\mathrm{Tr}(a)$. This yields

$$\tau(\Omega_{\mathbb{A}}) = 2\pi t c_1(\mathbb{B}) \ , \ h(\Omega_{\mathbb{A}} \wedge \Omega_{\mathbb{A}}) = -\mathrm{Tr}(\Omega_{\mathbb{B}} \wedge \Omega_{\mathbb{B}}) = 4\pi^2(c_1(\mathbb{B})^2 - 2c_2(\mathbb{B})) = 8\pi^2\mathrm{ch}_2 \ ,$$

where $\mathrm{ch}_2 := \frac{1}{2}c_1^2 - c_2$ stands for the second component of the Chern character. Theorem 3.2 gives

Proposition 3.3. *The natural Kähler form on the moduli space*

$$\mathcal{M}_t^* = \mathcal{M}_t^*(E, E_0, A_0)$$

of irreducible solutions of (V_t) *is given by*

$$\omega_{\mathcal{M}_t^*} = p_{\mathcal{M}_t^*,*}\left[-4\pi^2 \mathrm{ch}_2(\mathbb{B}) \wedge p_X^*(\omega_g^{n-1}) - 2t\pi(n-1)! c_1(\mathbb{B}) \wedge p_X^*(\mathrm{vol}_g)\right], \tag{25}$$

where $\mathbb{B}$ *denotes the universal connection on the universal bundle* $\mathbb{E} := \mathcal{C}^* \times E/\mathcal{G}$ *on* $\mathcal{M}_t^* \times X$.

Remark 3.4. *Let* $\mathcal{L}_0$ *be a holomorphic line bundle on* X *of Chern class* $l_0 \in \mathrm{NS}(X)$, χ_0 *a Hermite-Einstein metric on* $\mathcal{L}_0$, a_0 *the corresponding Hermite-Einstein connection on the Hermitian line bundle* (L_0, χ_0), *and* γ_0 *the first Chern form of* a_0. *Then the moduli spaces* $\mathcal{M}_t^*(E, E_0, A_0)$ *and* $\mathcal{M}^*_{t-2\pi\Lambda\gamma_0}(E \otimes L_0, E_0 \otimes L_0, A_0 \otimes a_0)$ *can be identified via the map* $[B, s] \mapsto [B \otimes a_0, s \otimes \mathrm{id}_{L_0}]$ *and the Kähler forms given by formula (25) correspond via this identification.*

Proof. Let $\mathcal{C}^*$, $\tilde{\mathcal{C}}^*$ be the configuration spaces of irreducible pairs associated with the two data systems (E, E_0), $(E \otimes L_0, E_0 \otimes L_0)$ respectively, $\mathcal{B}^*$, $\tilde{\mathcal{B}}^*$ the corresponding quotients, $\mathbb{E} \to \mathcal{B}^* \times X$, $\tilde{\mathbb{E}} \to \tilde{\mathcal{B}}^* \times X$ the corresponding universal bundles, and $\mathbb{B}$, $\tilde{\mathbb{B}}$ the corresponding universal connections. Using the formula

$$i\Lambda F_{B \otimes a_0} = i\Lambda F_B + 2\pi\Lambda\gamma_0$$

we see that the map $j : \mathcal{C}^* \to \tilde{\mathcal{C}}^*$ given by $j(B, s) = (B \otimes a_0, s \otimes \mathrm{id}_{L_0})$ maps the space of solutions of the equation $(V_t(E, E_0, A_0))$ bijectively onto the space of solutions of the equation $(V_{t-2\pi\Lambda\gamma_0}(E \otimes L_0, E_0 \otimes L_0, A_0 \otimes a_0))$. We have

$$([j] \times \mathrm{id}_X)^*(\tilde{\mathbb{E}}) = \mathbb{E} \otimes p_X^*(L_0)\ ,\ ([j] \times \mathrm{id}_X)^*(\tilde{\mathbb{B}}) = \mathbb{B} \otimes p_X^*(a_0)\ ,$$

$$([j] \times \mathrm{id}_X)^*(c_1(\tilde{\mathbb{B}})) = c_1(\mathbb{B}) + r p_X^*(\gamma_0)\ ,$$

$$([j] \times \mathrm{id}_X)^*(2c_2(\tilde{\mathbb{B}}) - c_1^2(\tilde{\mathbb{B}})) = (2c_2(\mathbb{B}) - c_1^2(\mathbb{B})) - 2c_1(\mathbb{B}) \wedge p_X^*(\gamma_0) - r p_X^*(\gamma_0^2)\ .$$

The statement follows now by direct computation. ∎

Let $\bar{\mathcal{A}}(E)$ be the space of semi-connections on E, $\bar{\mathcal{C}}^s := [\bar{\mathcal{A}}(E) \times A^0(\mathrm{Hom}(E, E_0))]^s$ the space of *simple pairs of type* $(E, \mathcal{E}_0)$ endowed with the natural action of the complex gauge group $\mathcal{G}^{\mathbb{C}} := \Gamma(X, \mathrm{GL}(E))$, and $\bar{\mathcal{B}}^s$ the quotient

$$\bar{\mathcal{B}}^s := {\bar{\mathcal{C}}^s}\big/{\mathcal{G}^{\mathbb{C}}}$$

(see [OT2] p. 559 for details). After suitable Sobolev completions (which will again be omitted) this quotient becomes an infinite dimensional complex space. As in the real gauge theoretical framework we obtain a universal bundle

$$\mathscr{E} := {\bar{\mathcal{C}}^s \times E}\big/{\mathcal{G}^{\mathbb{C}}}$$

of rank r on the product $\bar{\mathcal{B}}^s \times X$. Fixing a holomorphic structure $\mathcal{E}_0$ on E_0, we get a finite dimensional complex subspace

$$\mathcal{M}^s = \mathcal{M}^s(E, \mathcal{E}_0) \subset \bar{\mathcal{B}}^s\ ,$$

called the moduli space of simple holomorphic pairs of type $(E, \mathcal{E}_0)$ and the restriction of $\mathscr{E}$ to $\mathcal{M}^s \times X$ is holomorphic.

Choosing $\mathcal{E}_0$ to be the holomorphic structure on E_0 defined by the semi-connection $\bar\partial_{A_0}$, we obtain a map $KH : \mathcal{B}^* \to \bar{\mathcal{B}}^s$ induced by $(B, s) \mapsto (\mathcal{E}_{\bar\partial_B}, s : \mathcal{E}_{\bar\partial_B} \to \mathcal{E}_0)$.

Note that one has an obvious isomorphism

$$(KH \times \mathrm{id}_X)^*(\mathscr{E}) \simeq \mathbb{E} \tag{26}$$

and an obvious inclusion $KH(\mathcal{M}_t^*) \subset \mathcal{M}^s$. The image of this inclusion is given by the Kobayashi-Hitchin correspondence for twisted holomorphic pairs (see Theorem 2.7 [OT2]):

Theorem 3.5. *The map $(B, s) \mapsto (\mathcal{E}_{\bar\partial_B}, s : \mathcal{E}_{\bar\partial_B} \to \mathcal{E}_0)$ induces a real analytic isomorphism*

$$KH : \mathcal{M}_t^*(E, E_0, A_0) \xrightarrow{\simeq} \mathcal{M}_{\mathfrak{t}}^{\mathrm{st}}(E, \mathcal{E}_0)$$

onto the moduli space of $\mathfrak{t}$-stable pairs of type $(E, \mathcal{E}_0)$ with $\mathfrak{t} := \frac{(n-1)!\mathrm{Vol}_g(X)}{2\pi} t$.

In order to save on notations we will denote by the same symbol ω_t the Kähler form $\omega_{\mathcal{M}_t^*}$ defined above and also its image under the isomorphism KH.

Let $G \to \mathrm{Aut}(\mathcal{E}_0)$ be a morphism of Lie groups. Then G acts naturally on the space $\bar{\mathcal{A}}(E) \times A^0(\mathrm{Hom}(E, E_0))$ of pairs of type $(E, \mathcal{E}_0)$ leaving invariant the holomorphy, the simplicity, and the $\mathfrak{t}$-stability conditions. Since this action commutes with the action of the complex gauge group $\mathcal{G}^{\mathbb{C}}$, it induces a G-action on $\bar{\mathcal{B}}^s$ leaving invariant the finite dimensional subspaces $\mathcal{M}^s$ and $\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}}(E, \mathcal{E}_0)$.

Proposition 3.3 combined with the Kobayashi-Hitchin correspondence given by Theorem 3.5 and the isomorphism (26) has an important consequence: the Kähler class ω_t has a natural lift in equivariant cohomology.

Theorem 3.6. *For any morphism of Lie groups $G \to \mathrm{Aut}(\mathcal{E}_0)$ the class*

$$\left\{-4\pi^2 \mathrm{ch}_2^G(\mathscr{E}) \cup p_X^*[\omega_g^{n-1}] - 2t\pi(n-1)! c_1^G(\mathscr{E}) \cup p_X^*[\mathrm{vol}_g]\right\} / [X]$$

is a lift of the Kähler class $[\omega_t]$ to $H_G^2(\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}}(E, \mathcal{E}_0), \mathbb{R})$.

Proof. Put $\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}} := \mathcal{M}_{\mathfrak{t}}^{\mathrm{st}}(E, \mathcal{E}_0)$. Using the trivial action on the second factor, the product $\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}} \times X$ becomes a G-space, and one has natural isomorphisms

$$\begin{aligned} H_G^*(\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}} \times X, \mathbb{R}) &\simeq H^*(EG \times_G (\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}} \times X), \mathbb{R}) \simeq H^*((EG \times_G \mathcal{M}_{\mathfrak{t}}^{\mathrm{st}}) \times X, \mathbb{R}) \\ &\simeq H_G^*(\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}}, \mathbb{R}) \otimes H^*(X, \mathbb{R}) \ . \end{aligned}$$

Therefore the slant product with the fundamental class $[X]$ is well defined on both cohomology algebras $H^*(\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}} \times X, \mathbb{R})$, $H_G^*(\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}} \times X, \mathbb{R})$, and defines a commutative diagram

$$\begin{tikzcd} H_G^j(\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}} \times X, \mathbb{R}) \arrow[r] \arrow[d, "/[X]"'] & H^j(\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}} \times X, \mathbb{R}) \arrow[d, "/[X]"'] \\ H_G^{j-2n}(\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}}, \mathbb{R}) \arrow[r] & H^{j-2n}(\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}}, \mathbb{R}) \end{tikzcd}$$

for any $j \in \mathbb{N}$. Since the equivariant Chern classes $c_k^G(\mathscr{E})$ are lifts of the classes $c_k(\mathscr{E})$ to equivariant cohomology, and fibre integration $p_{\mathcal{M}_t^*,*}$ induces the morphism $/[X]$ in de Rham cohomology, the result follows from formula (25).

■

In [OT2] we have also shown that, for sufficiently large $\mathfrak{t}$, the moduli space $\mathcal{M}_{\mathfrak{t}}^{\mathrm{st}}(E, \mathcal{E}_0)$ can be identified with the Quot space $\mathcal{Q}uot_{\mathcal{E}_0}^E$ of quotients of $\mathcal{E}_0$ with locally free kernel of $\mathcal{C}^\infty$-type E.

Remark 3.7. *The natural identification $\mathcal{M}^{\rm st}_{\mathfrak{t}}(E,\mathcal{E}_0)=\mathcal{Q}uot^E_{\mathcal{E}_0}$ (for $\mathfrak{t}\gg 0$) is equivariant with respect to the* $\mathrm{Aut}(\mathcal{E}_0)$*-actions on the two spaces. Via this identification the restriction of the universal bundle $\mathscr{E}$ to $\mathcal{Q}uot^E_{\mathcal{E}_0}\times X$ coincides (as an* $\mathrm{Aut}(\mathcal{E}_0)$*-bundle) with the universal kernel associated with this Quot space.*

In order to save on notations we denote the restrictions of the universal bundles $\mathscr{E}$ and $\mathbb{E}$ to $\mathcal{M}^{\rm st}_{\mathfrak{t}}(E,\mathcal{E}_0)\times X$ and $\mathcal{M}^*_t(E,E_0,A_0)\times X$ respectively by the same symbols. Using Theorem 3.5 and the isomorphism (26) we obtain

Remark 3.8. *Suppose that t is sufficiently large such that the isomorphism KH defines a real analytic isomorphism*

$$KH:\mathcal{M}^*_t(E,E_0,A_0)\to \mathcal{Q}uot^E_{\mathcal{E}_0}\ .$$

Via this isomorphism the underlying $\mathcal{C}^\infty$-bundle of the pull-pack of the universal kernel $\mathscr{E}$ on $\mathcal{Q}uot^E_{\mathcal{E}_0}\times X$ can be identified with the universal bundle $\mathbb{E}$ on $\mathcal{M}^*_t(E,E_0,A_0)\times X$. In the case $r=1$ the identification $\mathcal{M}^{\rm st}_{\mathfrak{t}}(E,\mathcal{E}_0)=\mathcal{Q}uot^E_{\mathcal{E}_0}$ holds for $\mathfrak{t}>-\frac{\deg_g(E)}{\mathrm{rk}(E)}$ [OT2].

The Künneth decomposition of $c_1(\mathbb{E})\in H^2(\mathcal{B}^*\times X)$ has the form

$$c_1(\mathbb{E})=-\gamma\otimes 1+\boldsymbol{\delta}+1\otimes c_1(E)\ , \tag{27}$$

where $\gamma=-c_1(\mathbb{E}|_{\mathcal{B}^*\times\{x_0\}})$ for a fixed point $x_0\in X$, and $\delta\in H^1(\mathcal{B}^*,\mathbb{Z})\otimes H^1(X,\mathbb{Z})$. We will denote by the same symbol $\boldsymbol{\delta}$ the corresponding morphism $H_1(X,\mathbb{Z})/\mathrm{Tors}\to H^1(\mathcal{B}^*,\mathbb{Z})$. Let (h_i) be a basis of the free $\mathbb{Z}$-module $H_1(X,\mathbb{Z})/\mathrm{Tors}$, (h^i) the dual basis of $H^1(X,\mathbb{Z})$ and

$$h_{ij}:=\langle h^i\cup h^j\cup[\omega_g^{n-1}],[X]\rangle\ .$$

Then

$$[p_{\mathcal{B}^*}]_*(c_1^2(\mathbb{E})\cup p_X^*([\omega_g^{n-1}]))=-2\boldsymbol{\theta}-2\mathrm{deg}_g(E)\gamma\ , \tag{28}$$

where

$$\boldsymbol{\theta}:=\sum_{i<j}h_{ij}\ \boldsymbol{\delta}(h_i)\cup\boldsymbol{\delta}(h_j)\in H^2(\mathcal{B}^*,\mathbb{Z})\ .$$

With this notations, putting $\alpha:=p_{\mathcal{B}^*}(c_2(\mathbb{E}))\in H^2(\mathcal{B}^*,\mathbb{Z})$, we get a more explicit formula for the Kähler class $[\omega_{\mathcal{M}^*_t}]$

$$[\omega_{\mathcal{M}^*_t}]=\left\{4\pi^2\big(\alpha+\boldsymbol{\theta}+\deg_g(E)\gamma\big)+2t\pi(n-1)!\mathrm{Vol}_g(X)\gamma\right\}\big|_{\mathcal{M}^*_t}\ . \tag{29}$$

The class $\boldsymbol{\theta}$ can be written as pull-back of a tautological cohomology class θ on the moduli space $\mathcal{B}(L)$ of gauge equivalence classes of Hermitian connections on the Hermitian line bundle $l:=\det(E)$, and the restriction $\boldsymbol{\theta}|_{\mathcal{M}^*_t}$ coincides with the pull back of the classical theta class of $\mathrm{Pic}^{c_1(E)}(X)$. Indeed, recall that $\mathcal{B}(L)$ can be identified with the total space of a vector bundle over the torus of Yang-Mills connections on E (which can be identified via the classical Kobayashi-Hitchin correspondence with $\mathrm{Pic}^{c_1(E)}(X)$), hence the embedding $\mathrm{Pic}^{c_1(E)}(X)\hookrightarrow\mathcal{B}(L)$ is a homotopical equivalence. Therefore we have well defined *isomorphism*

$$\delta:H_1(X,\mathbb{Z})/\mathrm{Tors}\to H^1(\mathcal{B}(L),\mathbb{Z})\ ,$$

which defines an element (denoted by the same symbol) of $H^1(\mathcal{B}(E),\mathbb{Z})\otimes H^1(X,\mathbb{Z})$. Denoting by ν the natural map $\mathcal{B}^*\to\mathcal{B}(L)$ given by $\nu([A,\varphi])):=[\det(A)]$ we have

$$\boldsymbol{\delta}=(\nu\times\mathrm{id}_X)^*(\delta)\ ,\ \boldsymbol{\theta}=\nu^*(\theta)\ .$$

In the case $r=1$ formula (29) becomes

$$\frac{1}{4\pi^2}[\omega_{\mathcal{M}_t^*}]=\theta+(\deg_g(E)+\mathfrak{t})\gamma\ , \tag{30}$$

which specializes to a formula obtained by Perutz (see Theorem 3 [P]) for $r_0=1$ and $n=1$.

3.2. **Volumina of Abelian Quot spaces.** Let X be compact complex manifold, let $\mathcal{E}_0$ be a locally free sheaf of rank r_0 on X, and let E be a differentiable vector bundle of rank r. As above we denote by $\mathcal{Q}uot^E_{\mathcal{E}_0}$ the Quot space of quotients $q:\mathcal{E}_0\to Q$ of $\mathcal{E}_0$ with locally free kernel $\mathcal{C}^\infty$-isomorphic to E. Since locally-freeness is an open condition in flat families it follows that $\mathcal{Q}uot^E_{\mathcal{E}_0}$ is an open subspace of the Douady space $\mathcal{Q}uot_{\mathcal{E}_0}$. This Quot space is characterized by obvious universal properties.

For $r=1$ the bundle E is a differentiable line bundle, hence its differentiable isomorphism type is determined by its Chern class. Therefore in this case it is convenient to fix $m\in\mathrm{NS}(X)$ and to denote by $\mathcal{Q}uot^m_{\mathcal{E}_0}$ the Quot space of quotients $q:\mathcal{E}_0\to Q$ of $\mathcal{E}_0$ with locally free kernel of rank 1 and Chern class m. We will give an explicit description of certain Quot spaces of this type.

3.2.1. *Symmetric powers of curves.* With the notations and under the conditions above suppose that $r=r_0=1$ and X is a curve. In this case the Quot space $\mathcal{Q}uot^E_{\mathcal{E}_0}$ can be identified with the symmetric power $X^{(d)}$ and, via this identification, the universal kernel $\mathscr{E}$ and the universal quotient $\mathscr{Q}$ on $\mathcal{Q}uot^E_{\mathcal{E}_0}\times X$ can be identified with $\mathcal{O}(-\Delta)\otimes p_X^*(\mathcal{E}_0)$ and $p_X^*(\mathcal{E}_0)\otimes\mathcal{O}_\Delta$ respectively. Here Δ is the tautological divisor of $X^{(d)}\times X$. By Proposition 3.14 we obtain a canonical isomorphism

$$\mathcal{T}_{\mathcal{Q}uot^E_{\mathcal{E}_0}}\stackrel{\simeq}{\longrightarrow}\rho_*(\mathcal{O}(\Delta)_\Delta)\ . \tag{31}$$

By formula (27) the Künneth decomposition of $c_1(\mathscr{E})$ reads

$$c_1(\mathscr{E})=-\gamma\otimes 1+\delta+\deg(E)\otimes\{X\}\ , \tag{32}$$

where $\{X\}\in H^2(X,\mathbb{Z})$ stands for the cohomological fundamental class of X and we denote by the same symbol the pull-back of δ via the map

$$\nu\times\mathrm{id}_X:\mathcal{Q}uot^E_{\mathcal{E}_0}\times X\to\mathrm{Pic}^{\deg(E)}(X)\times X\ .$$

Furthermore, formula (28) gives

$$c_1^2(\mathscr{E})/[X]=-2\theta-2\deg(E)\gamma\ , \tag{33}$$

where we use the same symbol for the pull-back of $\theta\in H^2(\mathrm{Pic}^{\deg(E)}(X),\mathbb{Z})$ via the morphism $\nu:\mathcal{Q}uot^E_{\mathcal{E}_0}\to\mathrm{Pic}^{\deg(E)}(X)$.

The numbers obtained by evaluating the classes $\gamma^{d-j}\theta^j$ on the fundamental class of $\mathcal{Q}uot^E_{\mathcal{E}_0}$ can be computed explicitly, and the result is:

$$\langle\gamma^{d-j}\theta^j,[\mathcal{Q}uot^E_{\mathcal{E}_0}]\rangle=\begin{cases}\frac{\mathfrak{g}!}{(\mathfrak{g}-j)!} & \text{for}\quad 0\leq j\leq\mathfrak{g}\\ 0 & \text{for}\quad j>\mathfrak{g}\end{cases} \tag{34}$$

This is the classical Poincaré formula (see for instance [ACGH] p. 343), but can also be obtained as a special case of Theorem 3.8 in [OT2] by interpreting these numbers as gauge theoretical Gromov-Witten invariants. Using this formula and our formula (30) for the Kähler class $[\omega_t]$ on Abelian moduli spaces, we obtain for the volume of $\mathcal{Q}uot^E_{\mathcal{E}_0}$ the formula:

$$V_t(\mathcal{Q}uot^E_{\mathcal{E}_0}) = (4\pi^2)^d \sum_{i=0}^{\min(d,\mathfrak{g})} \binom{\mathfrak{g}}{i} \frac{1}{(d-i)!}(\deg(E)+\mathfrak{t})^{d-i}$$

Up to the normalization factor π^d it specializes to the Manton-Nasir formula [MN] for the volume of $X^{(d)}$ when $\mathcal{E}_0 = \mathcal{O}_X$ and $t = \frac{1}{2}$.

3.2.2. *The acyclic case.* We begin with a simple flatness criterion:

Lemma 3.9. *Let $\pi : \mathcal{X} \to B$ be a morphism of complex spaces, and let $\mathcal{F} \xrightarrow{j} \mathcal{E}$ be a morphism of coherent sheaves on $\mathcal{X}$ such that $\mathcal{E}$ is flat over B. Then the following two conditions are equivalent:*

(i) For every $b \in B$ the restriction $j_b : \mathcal{F}_{X_b} \to \mathcal{E}_{X_b}$ of j to the fibre X_b of π is a sheaf monomorphism.
(ii) j is a monomorphism and the quotient $\mathcal{Q} := \mathcal{E}/j(\mathcal{F})$ is flat over B.

Proof. For $b \in B$ and $x \in X_b$ the natural morphism

$$j^b_x : \mathcal{F}_x \otimes_{\mathcal{O}_b} \mathbb{C} \to \mathcal{E}_x \otimes_{\mathcal{O}_b} \mathbb{C}$$

induced by j is just the morphism induced by j_b between these stalks. Therefore (i) is equivalent to the injectivity of j^b_x for every $b \in B$, $x \in X_b$. Fix $b \in B$, $x \in X_b$ and note that $\mathcal{E}_x$ is a flat $\mathcal{O}_b$-module by assumption. By Corollary 1.5 p. 165 [BaSt] j^b_x is a monomorphism if and only if $j_x : \mathcal{F}_x \to \mathcal{E}_x$ is a monomorphism and $\operatorname{coker}(j_x)$ is a flat $\mathcal{O}_b$-module.

■

Remark 3.10. *The hypothesis of Lemma 3.9 and condition (i) of this lemma are satisfied when π is a flat morphism, $\mathcal{F}$, $\mathcal{E}$ are locally free, and for any $b \in B$ the following two conditions hold:*

(i) the fibre X_b is smooth,
(ii) the restriction $j_b : \mathcal{F}_{X_b} \to \mathcal{E}_{X_b}$ is a generically injective morphism of vector bundles over X_b.

Therefore, by Lemma 3.9, in this case $\mathcal{Q} := \mathcal{E}/j(\mathcal{F})$ is flat over B.

Note that the statement holds even if the total space $\mathcal{X}$ is not reduced.

Now we come back to our situation: let X be a compact complex manifold.

Definition 3.11. *A pair $(m, \mathcal{E}_0)$ consisting of a class $m \in \mathrm{NS}(X)$ and a holomorphic bundle $\mathcal{E}_0$ of rank r_0 on X is called acyclic if*

$$h^i(\mathcal{L}^\vee \otimes \mathcal{E}_0) = 0 \ \ \forall i > 0 \ , \ \forall [\mathcal{L}] \in \mathrm{Pic}^m(X) \ .$$

Examples: 1. Suppose that X is a smooth projective manifold, H an ample divisor on X, and $(m, \mathcal{E}_0)$ is a pair consisting of a class $m \in \mathrm{NS}(X)$ and a holomorphic vector bundle $\mathcal{E}_0$. Using Serre's vanishing theorem and the compactness of $\mathrm{Pic}^m(X)$ one shows easily that for all sufficiently large $k \in \mathbb{N}$ the pair $(m, \mathcal{E}_0(kH))$ is acyclic.

2. Suppose now that X is a curve, $m \in \mathbb{Z}$ and $\mathcal{E}_0$ is a polystable bundle on X with $\deg(\mathcal{E}_0) > r_0 m + 2r_0(g(X) - 1)$. Then for every $[\mathcal{L}] \in \mathrm{Pic}^m(X)$ the bundle

$\mathcal{K}\otimes\mathcal{E}_0^\vee\otimes\mathcal{L}$ admits a Hermite-Einstein metric with negative Einstein constant, so that $H^0(\mathcal{K}\otimes\mathcal{E}_0^\vee\otimes\mathcal{L})=0$. Using Serre duality we see that the pair $(m,\mathcal{E}_0)$ is acyclic.

Let $\mathscr{L}_m$ be a Poincaré line bundle on $\mathrm{Pic}^m(X)\times X$ normalized with respect to $x_0\in X$. Denote by π, p the projections

$$\pi:\mathrm{Pic}^m(X)\times X\to\mathrm{Pic}^m(X)\ ,\ p:\mathrm{Pic}^m(X)\times X\to X\ .$$

When the pair $(m,\mathcal{E}_0)$ is acyclic $\mathcal{Q}uot^m_{\mathcal{E}_0}$ can be identified with a projective bundle over $\mathrm{Pic}^m(X)$. The proof is based on the universal property of a Quot space.

Proposition 3.12. *Suppose the pair $(m,\mathcal{E}_0)$ is acyclic.*

(i) The sheaf $\mathscr{V}:=R^0\pi_*(\mathscr{L}_m^\vee\otimes p^*(\mathcal{E}_0))$ on $\mathrm{Pic}^m(X)$ is locally free.

(ii) $\mathrm{ch}(\mathscr{V})=\pi_*\big(\mathrm{ch}(\mathscr{L}_m^\vee\otimes p^*(\mathcal{E}_0))\cup p^*(\mathrm{td}(X))\big)$.

(iii) Let $\nu:\mathbb{P}(\mathscr{V})\to\mathrm{Pic}^m(X)$, $p_X:\mathbb{P}(\mathscr{V})\times X\to X$, $\rho:\mathbb{P}(\mathscr{V})\times X\to\mathbb{P}(\mathscr{V})$ be the natural projections, and let

$$\mathscr{E}:=(\nu\times\mathrm{id})^*(\mathscr{L}_m)\otimes\rho^*(\mathcal{O}_{\mathscr{V}}(-1))\ .$$

There exists a tautological monomorphism $j:\mathscr{E}\to p_X^*(\mathcal{E}_0)$ such that the corresponding quotient $\mathscr{Q}$ of $p_X^*(\mathcal{E}_0)$ is flat over $\mathbb{P}(\mathscr{V})$.

(iv) The obtained epimorphism $u:p_X^*(\mathcal{E}_0)\to\mathscr{Q}$ defines an isomorphism

$$\mathbb{P}(\mathscr{V})\xrightarrow{\simeq}\mathcal{Q}uot^m_{\mathcal{E}_0}\ .$$

Proof. Since the pair $(m,\mathcal{E}_0)$ is acyclic we have $h^0(\mathcal{L}^\vee\otimes\mathcal{E}_0)=\chi(\mathcal{L}^\vee\otimes\mathcal{E}_0)$, which is a topological invariant of the pair $(m,\mathcal{E}_0)$, hence it is constant with respect to $[\mathcal{L}]\in\mathrm{Pic}^m(X)$. The first statement follows from Grauert's locally freeness theorem. The second statement is a consequence of the Grothendieck-Riemann-Roch theorem for families taking into account that $R^i\pi_*(\mathscr{L}_m^\vee\otimes p^*(\mathcal{E}_0))=0$ for $i>0$. For the third statement note that the evaluation morphism

$$\mathrm{ev}:\mathscr{L}_m\otimes\pi^*(\mathscr{V})\to p^*(\mathcal{E}_0)$$

on $\mathrm{Pic}^m(X)\times X$ can be regarded as an element of

$$H^0(\mathscr{L}_m^\vee\otimes\pi^*(\mathscr{V})^\vee\otimes p^*(\mathcal{E}_0))=H^0\left((\nu\times\mathrm{id}_X)_*\big((\nu\times\mathrm{id}_X)^*(\mathscr{L}_m^\vee)\otimes\mathcal{O}_{\mathscr{V}}(1)\otimes p_X^*(\mathcal{E}_0)\big)\right)$$
$$=H^0\big(\mathcal{H}om\left((\nu\times\mathrm{id}_X)^*(\mathscr{L}_m)\otimes\mathcal{O}_{\mathscr{V}}(-1),p_X^*(\mathcal{E}_0)\right)\big)\ ,$$

hence it defines a morphism $(\nu\times\mathrm{id}_X)^*(\mathscr{L}_m)\otimes\mathcal{O}_{\mathscr{V}}(-1)\to p_X^*(\mathcal{E}_0)$ of sheaves on $\mathbb{P}(\mathscr{V})\times X$. Using Remark 3.10 and Lemma 3.9 it follows that the corresponding morphism

$$j:(\nu\times\mathrm{id}_X)^*(\mathscr{L}_m)\otimes\mathcal{O}_{\mathscr{V}}(-1)\to p_X^*(\mathcal{E}_0)$$

is a sheaf monomorphism and the corresponding quotient $\mathscr{Q}$ is flat over $\mathbb{P}(\mathscr{V})$.

The fourth statement is a consequence of the third. It suffices to prove that the quotient epimorphism $p_X^*(\mathcal{E}_0)\to\mathscr{Q}$ satisfies the universal property of the tautological quotient over $\mathcal{Q}uot^m_{\mathcal{E}_0}$.

Let Y be an arbitrary complex space and $q_X:Y\times X\to X$, $q_Y:Y\times X\to Y$ the projections on the two factors. Let $v:q_X^*(\mathcal{E}_0)\to\mathcal{Q}$ be an epimorphism of coherent sheaves on $Y\times X$, such that $\mathcal{Q}$ is flat over Y and the kernel $\mathcal{F}:=\ker(v)$ is fibrewise locally free of rank 1 with Chern class m. Since $\mathcal{Q}$ is flat over Y it follows that $\mathcal{F}$ is flat over Y as well. Therefore $\mathcal{F}$ is a line bundle over $Y\times X$ and, using the universal property of the Picard group, there exists a morphism $\alpha : Y \to \mathrm{Pic}^m(X)$, a line bundle $\mathcal{M}$ on Y and a line bundle isomorphism

$$\beta : \mathcal{F} \to q_Y^*(\mathcal{M}) \otimes (\alpha \times \mathrm{id}_X)^*(\mathscr{L}_m)$$

over $Y \times X$. Since the base change property holds for $R^0\pi_*(\mathscr{L}_m^\vee \otimes p^*(\mathcal{E}_0))$, we obtain isomorphisms

$$\alpha^*(\mathscr{V}) \simeq (q_Y)_*((\alpha \times \mathrm{id})^*(\mathscr{L}_m^\vee \otimes p^*(\mathcal{E}_0)) \simeq (q_Y)_*((\alpha \times \mathrm{id})^*(\mathscr{L}_m)^\vee \otimes q_X^*(\mathcal{E}_0)) \ .$$

On the other hand, the monomorphism $\iota : \mathcal{F} \hookrightarrow q_X^*(\mathcal{E}_0)$ defines a section σ in $\mathcal{F}^\vee \otimes q_X^*(\mathcal{E}_0)$ which is fibrewise non-trivial by Lemma 3.9. Using the isomorphism β we obtain a fibrewise non-trivial section

$$s \in H^0\left(q_Y^*(\mathcal{M})^\vee \otimes (\alpha \times \mathrm{id})^*(\mathscr{L}_m)^\vee \otimes q_X^*(\mathcal{E}_0)\right) \ ,$$

hence a nowhere vanishing section $(q_Y)_*(s)$ in

$$\begin{aligned}(q_Y)_*(q_Y^*(\mathcal{M})^\vee \otimes (\alpha \times \mathrm{id})^*(\mathscr{L}_m)^\vee \otimes q_X^*(\mathcal{E}_0)) &= \mathcal{M}^\vee \otimes (q_Y)_*\left((\alpha \times \mathrm{id})^*(\mathscr{L}_m)^\vee \otimes q_X^*(\mathcal{E}_0)\right) \\ &= \mathcal{M}^\vee \otimes \alpha^*(\mathscr{V}) \ .\end{aligned}$$

Therefore $(q_Y)_*(s)$ identifies $\mathcal{M}$ with a line subbundle of $\alpha^*(\mathscr{V})$. This line subbundle induces a morphism $\lambda : Y \to \mathbb{P}(\mathscr{V})$ with $\nu \circ \lambda = \alpha$. It is now easy to see that the epimorphism v can be identified with the pull-back of u via λ, and that λ is the unique morphism $Y \to \mathbb{P}(\mathscr{V})$ which induces such an identification.

■

Let again (h_i) be a basis of $H_1(X, \mathbb{Z})/\mathrm{Tors}$, and let $\delta : H_1(X, \mathbb{Z})/\mathrm{Tors} \to H^1(\mathrm{Pic}^m(X), \mathbb{Z})$ be the natural isomorphism. Put $\lambda_i := \delta(h_i) \in H^1(\mathrm{Pic}^m(X), \mathbb{Z})$. Then one has $\delta = \sum_i \lambda_i \otimes h^i$. Now put $h^I := h^{i_1} \cup \cdots \cup h^{i_k}$, $\lambda_I := \lambda_{i_1} \cup \cdots \cup \lambda_{i_k} \in H^k(\mathrm{Pic}^m(X), \mathbb{Z})$. Then a simple computation shows that

$$\delta^k = (-1)^{\frac{k(k-1)}{2}} \sum_{\substack{I \subset \{1, \ldots b\} \\ |I| = k}} \lambda_I \otimes h^I \ .$$

Therefore, for any class $\mathfrak{c} \in H^{2n-k}(X, \mathbb{Z})$ one can write

$$\begin{aligned}\pi_*(\delta^k \cup p^*(\mathfrak{c})) &= (-1)^{\frac{k(k-1)}{2}} \pi_* \left\{ \sum_{\substack{I \subset \{1, \ldots b\} \\ |I| = k}} \lambda_I \otimes (h^I \cup \mathfrak{c}) \right\} \\ &= (-1)^{\frac{k(k-1)}{2}} \sum_{\substack{I \subset \{1, \ldots b\} \\ |I| = k}} \langle h^I \cup \mathfrak{c}, [X] \rangle \lambda_I \ .\end{aligned}$$

In particular

$$\pi_*(\delta^2 \cup p^*[\omega_g^{n-1}]) = -\sum_{i,j} \langle h^i \cup h^j \cup [\omega_g^{n-1}], [X] \rangle = -2\theta \ .$$

Identifying $H^*(\mathrm{Pic}^m(X), \mathbb{Z}) = \mathrm{Alt}^*(H^1(X, \mathbb{Z}), \mathbb{Z})$ we have

$$\lambda_I(h^{j_1}, \ldots, h^{j_k}) = \epsilon_I^J \ ,$$

$$\pi_*(\delta^k \cup p^*(\mathfrak{c}))(h^{j_1}, \ldots, h^{j_k}) = (-1)^{\frac{k(k-1)}{2}} \sum_{\substack{I \subset \{1, \ldots b\} \\ |I| = k}} \langle h^I \cup \mathfrak{c}, [X] \rangle \epsilon_I^J \ .$$

Note that

$$\sum_{\substack{I\subset\{1,\dots b\}\\ |I|=k}} h^I\epsilon_I^J = k!\ h^J\ .$$

This proves:

Remark 3.13. *For a class $\mathfrak{c}\in H^{2n-k}(X,\mathbb{Z})$ define $\mathfrak{k}_{\mathfrak{c}}\in \mathrm{Alt}^k(H^1(X,\mathbb{Z}),\mathbb{Z})$ by*

$$\mathfrak{k}_{\mathfrak{c}}(x^1,\dots x^k):=\langle x_1\cup\dots\cup x_k\cup\mathfrak{c},[X]\rangle\ .$$

Using the canonical identification $\mathrm{Alt}^k(H^1(X,\mathbb{Z}),\mathbb{Z})=H^k(\mathrm{Pic}^m(X),\mathbb{Z})$ one has

$$\pi_*(\delta^k\cup p^*(\mathfrak{c}))=(-1)^{\frac{k(k-1)}{2}}k!\ \mathfrak{k}_{\mathfrak{c}}\ .$$

Let ω_t be the Kähler form induced on $\mathcal{Q}uot^m_{\mathcal{E}_0}$ via the Kobayashi-Hitchin correspondence KH. Using Remark 3.8 and formula (30) we obtain a formula for the cohomology class $[\omega_t]$ in terms of the classes γ and θ defined in section 3.1.

In our case, by Proposition 3.12 (iii) we get $\gamma=c_1(\mathcal{O}_{\mathscr{V}}(1))$, hence

$$\begin{aligned}[\omega_t]&=4\pi^2\big(\nu^*(\theta)+\mathfrak{m}\gamma\big)+2t\pi(n-1)!\mathrm{Vol}_g(X)\gamma\\ &=4\pi^2\nu^*(\theta)+\big(4\pi^2\mathfrak{m}+2t\pi(n-1)!\mathrm{Vol}_g(X)\big)\gamma\ ,\end{aligned}$$

with

$$\mathfrak{m}:=\langle m\cup[\omega_g]^{n-1},[X]\rangle=\deg_g(E)\ ,\ \theta=\sum_{i<j}h_{ij}\delta(h_i)\cup\delta(h_i)\in H^2(\mathrm{Pic}^m(X))\ ,$$

$$h_{ij}:=\langle h^i\cup h^j\cup[\omega^{n-1}],[X]\rangle\ .$$

Put

$$A:=4\pi^2,\ B_t:=4\pi^2\mathfrak{m}+2t\pi(n-1)!\mathrm{Vol}_g(X),\ R:=\mathrm{rk}(\mathscr{V}),\ q:=\frac{b_1(X)}{2},\ N:=q+R-1\ .$$

The volume V_t of the Kähler manifold $(\mathcal{Q}uot^m_{\mathcal{E}_0},\omega_t)$ is

$$V_t=\frac{1}{N!}\int_{\mathbb{P}(\mathscr{V})}[A\nu^*(\theta)+B_t\gamma]^N\ .$$

Projecting onto $\mathrm{Pic}^m(X)$ and using formula (4.3) of [ACGH] we get

$$\begin{aligned}V_t&=\frac{1}{N!}\sum_{R-1\le k\le N}\binom{N}{k}A^kB_t^{N-k}\Big\langle\theta^k\cup\nu_*(\gamma^{N-k}),[\mathrm{Pic}^m(X)]\Big\rangle\\ &=\frac{1}{N!}\sum_{R-1\le k\le N}\binom{N}{k}A^kB_t^{N-k}\Big\langle\theta^k\cup s_{q-k}(\mathscr{V}),[\mathrm{Pic}^m(X)]\Big\rangle\ ,\end{aligned}\tag{35}$$

where $s_j(\mathscr{V})$ denotes the j-th Segre class of $\mathscr{V}$. The classes $\mathrm{ch}_i(\mathscr{V})$ can be computed using Proposition 3.12 (ii) and the result is

$$\mathrm{ch}_i(\mathscr{V})=\pi_*\left(\frac{1}{(2i)!}\delta^{2i}\cup p^*\left(e^{-m}\cup\mathrm{ch}(\mathcal{E}_0)\mathrm{td}(X)\right)^{(n-i)}\right)\ .$$

Now decompose

$$\mathrm{ch}(\mathcal{E}_0)\mathrm{td}(X)=\sum C_i$$

with $C_i\in H^{2i}(X,\mathbb{Q})$. Then, using Remark 3.13 we obtain

$$\mathrm{ch}_i(\mathscr{V})=\frac{1}{(2i)!}\sum_{s=0}^{n-i}\frac{(-1)^s}{s!}\pi_*\left(\delta^{2i}\cup p^*\left(m^sC_{n-i-s}\right)\right)=\sum_{s=0}^{n-i}\frac{(-1)^{i+s}}{s!}\mathfrak{k}_{m^sC_{n-i-s}}\ .$$

Using formula (1.2) p. 156 in [ACGH] we see that the Chern polynomial of $\mathscr{V}$ is given by

$$1+\sum c_i(\mathscr{V})=\exp\left(\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}\mathrm{ch}_i(\mathscr{V})\right)\ ,$$

hence the Segre polynomial is

$$1+\sum s_i(\mathscr{V})=\exp\left(\sum_{i=1}^{\infty}\frac{(-1)^{i}}{i}\mathrm{ch}_i(\mathscr{V})\right)=\exp\left(\sum_{i=1}^{q}\sum_{s=0}^{n-i}\frac{(-1)^{s}}{is!}\mathfrak{k}_{m^sC_{n-i-s}}\right)\ .$$

Suppose now that the basis (h_i) of $H_1(X,\mathbb{Z})/\mathrm{Tors}$ was chosen such that the image in $H^1(X,\mathcal{O})$ of the dual basis (h^i) is compatible with the complex orientation. With this choice the linear form $\langle\cdot,[\mathrm{Pic}^m(X)]\rangle$ on $H^{2q}(\mathrm{Pic}^m(X),\mathbb{Z})$ corresponds to the linear form

$$\phi\to\phi(h^1,\dots,h^{2q})$$

on $\mathrm{Alt}^{2q}(H^1(X,\mathbb{Z}),\mathbb{Z})$. Using (35) and noting $A^kB_t^{N-k}=(4\pi^2)^N(\deg(E)+\mathfrak{t})^{N-k}$ we obtain

$$V_t=\frac{(4\pi^2)^N}{N!}\Bigg\{\Bigg(\sum_{k=R-1}^{N}\binom{N}{k}(\deg_g(E)+\mathfrak{t})^{N-k}\theta^k\Bigg)\wedge$$

$$\wedge\exp\left(\sum_{i=1}^{q}\sum_{s=0}^{n-i}\frac{(-1)^{s}}{is!}\mathfrak{k}_{m^sC_{n-i-s}}\right)\Bigg\}(h_1,\dots,h_{2q})\ .$$

This shows that V_t is a polynomial function in t whose coefficients are determined by the cohomology classes $[\omega_g]$, m, $\mathrm{ch}_i(\mathcal{E}_0)$, and $\mathrm{td}_i(X)$.

3.3. **Volumina of non-Abelian Quot spaces.** In this section we will compute the volume of a Quot space $\mathcal{Q}uot^E_{\mathcal{E}_0}$ in the special case when $r=r_0$ and the base X is a complex curve of genus $\mathfrak{g}$. Since X is a curve, $\mathcal{C}^\infty$ bundles of rank r_0 on X are classified up to isomorphism by their degree, hence the Quot space $\mathcal{Q}uot^E_{\mathcal{E}_0}$ is the space of coherent subsheaves $\mathcal{E}\hookrightarrow\mathcal{E}_0$ with $\mathrm{rk}(\mathcal{E})=r_0$ and $\deg(\mathcal{E})=\deg(E)$. An element $(\mathcal{E}\hookrightarrow\mathcal{E}_0)\in\mathcal{Q}uot^E_{\mathcal{E}_0}$ defines a sheaf monomorphism $\wedge^r(\mathcal{E})\hookrightarrow\wedge^r(\mathcal{E}_0)$, hence $\mathcal{Q}uot^E_{\mathcal{E}_0}=\emptyset$ for $\deg(\mathcal{E}_0)<\deg(E)$ and $\mathcal{Q}uot^E_{\mathcal{E}_0}$ reduces to the singleton $\{\mathrm{id}_{\mathcal{E}_0}\}$ when $\deg(\mathcal{E}_0)=\deg(E)$. We put $d:=\deg(\mathcal{E}_0)-\deg(E)$ and we suppose that $d\geq 0$.

Recall the following well known result:

Proposition 3.14. *Let X be a complex curve, $\mathcal{E}_0$ (respectively E) a holomorphic (respectively differentiable) vector bundle of rank r_0 on X. Denote by $\mathscr{E}$ and $\mathscr{Q}$ the universal kernel and the universal quotient on $\mathcal{Q}uot^E_{\mathcal{E}_0}\times X$. The space $\mathcal{Q}uot^E_{\mathcal{E}_0}$ is smooth of dimension rd, and there is a canonical identification*

$$\mathcal{T}_{\mathcal{Q}uot^E_{\mathcal{E}_0}}\xrightarrow{\simeq}\rho_*(\mathscr{E}^\vee\otimes\mathscr{Q})\ ,\tag{36}$$

where $\rho:\mathcal{Q}uot^E_{\mathcal{E}_0}\times X\to\mathcal{Q}uot^E_{\mathcal{E}_0}$ denotes the projection onto the first factor.

Proposition 3.15. *Suppose that X is a curve, $r=r_0$ and put $d:=\deg(\mathcal{E}_0)-\deg(E)$, $\gamma:=-c_1\big(\mathscr{E}\big|_{\mathcal{Q}uot^E_{\mathcal{E}_0}\times\{x_0\}}\big)$. Then $\gamma^{d+1}=0$.*

Proof. Let
$$0 \to \mathscr{E} \xrightarrow{\psi} p_X^*(\mathcal{E}_0) \to \mathscr{Q} \to 0$$
be the universal quotient on $\mathcal{Q}uot^E_{\mathcal{E}_0} \times X$. For a point $q \in \mathcal{Q}uot^E_{\mathcal{E}_0}$ we denote by $\mathcal{E}_q$, $\mathcal{Q}_q$ the sheaves on X defined by the restrictions of $\mathscr{E}$, $\mathscr{Q}$ on the fibre $\{q\} \times X$. Since $\mathscr{Q}$ is flat over $\mathcal{Q}uot^E_{\mathcal{E}_0}$, we see by Lemma 3.9 that for every $q \in \mathcal{Q}uot^E_{\mathcal{E}_0}$ the restriction $\psi_q : \mathcal{E}_q \to \mathcal{E}_0$ is a sheaf monomorphism; it coincides with the kernel of the quotient q. Similarly for every $x \in X$ denote by $\mathscr{E}^x$ the bundle on $\mathcal{Q}uot^E_{\mathcal{E}_0}$ defined by the restriction of $\mathscr{E}$ to $\mathcal{Q}uot^E_{\mathcal{E}_0} \times \{x\}$ and denote by $\psi^x : \mathscr{E}^x \to \underline{\mathcal{E}_0(x)}$ the morphism induced by ψ.

Since $\mathscr{E}$ and $p_X^*(\mathcal{E}_0)$ are locally free of the same rank one has
$$\begin{aligned}\mathrm{supp}(\mathscr{Q}) = Z(\det(\psi)) &= \{(q,x) \in \mathcal{Q}uot^E_{\mathcal{E}_0} \times X |\ \det(\psi_q)(x) = 0\}\\ &= \{(q,x) \in \mathcal{Q}uot^E_{\mathcal{E}_0} \times X |\ \det(\psi^x)(q) = 0\}\ .\end{aligned}$$
The second equality shows that for every $x \in X$ one has
$$Z(\det(\psi^x)) = \{q \in \mathcal{Q}uot^E_{\mathcal{E}_0} |\ \det(\psi_q)(x) = 0\} = \{q \in \mathcal{Q}uot^E_{\mathcal{E}_0} |\ x \in \mathrm{supp}(Q_q)\}\ .$$
Therefore, for $k > d$ pairwise distinct points $x_1, \dots, x_k \in X$ one has
$$\bigcap_{i=1}^{k} Z(\det(\psi^{x_i})) = \emptyset\ . \tag{37}$$
This implies
$$\gamma^k = \cup_{i=1}^k c_1\big(\det(\mathscr{E}^{x_i})^\vee \otimes \wedge^r \mathcal{E}_0(x_i)\big) = c_k\Big(\bigoplus_{i=1}^{k}\big(\det(\mathscr{E}^{x_i})^\vee \otimes \wedge^r \mathcal{E}_0(x_i)\big)\Big) = 0$$
■

Let now E_i, $\mathcal{L}_i$ ($i \in \{1,2\}$) be differentiable, respectively holomorphic line bundles on X, and let $\mathcal{Q}_i := \mathcal{Q}uot^{E_i}_{\mathcal{L}_i}$, $\mathscr{E}_i$ and $\mathscr{Q}_i$ be the associated Quot spaces, universal kernels and universal quotients. Put $l_i := \deg(\mathcal{L}_i)$, $m_i := \deg(E_i)$, $d_i := \deg(\mathcal{L}_i) - \deg(E_i)$.

Consider the product $\mathcal{Q}_{12} := \mathcal{Q}_1 \times \mathcal{Q}_2$ and denote by $\rho_{12} : \mathcal{Q}_{12} \times X \to \mathcal{Q}_{12}$, $p_i : \mathcal{Q}_{12} \to \mathcal{Q}_i$, $q_i := p_i \times \mathrm{id}_X : \mathcal{Q}_{12} \times X \to \mathcal{Q}_i \times X$ the natural projections. For later computations we will need the characteristic classes of the push-forward:
$$\mathcal{N}_{12} := (\rho_{12})_* \{q_1^*(\mathscr{E}_1)^\vee \otimes q_2^*(\mathscr{Q}_2)\}\ ,$$
which is a locally free sheaf of rank d_2 on $\mathcal{Q}_{12}$. Since the higher direct images of the right hand sheaf on $\mathcal{Q}_{12}$ vanish, we obtain by the Grothendieck-Riemann-Roch theorem
$$\mathrm{ch}(\mathcal{N}_{12}) = (-\mathfrak{g} + l_2 - m_1 - \theta_1)e^{\gamma_1} + (\mathfrak{g} + m_1 - m_2 + (\theta_1 + \theta_2 + \sigma_{12}))e^{\gamma_1 - \gamma_2}\ , \tag{38}$$
where θ_i, γ_i stand for the pull-backs of the corresponding classes on $\mathcal{Q}_i$ and
$$\sigma_{12} := (\rho_{12})_*(q_1^*(\delta_1) \cup q_2^*(\delta_2))\ .$$
Since $\mathcal{Q}_{12}$ has torsion-free integral cohomology, formula (38) implies
$$c_t(\mathcal{N}_{12}) = \frac{(1 + t\gamma_1)^{-\mathfrak{g} + l_2 - m_1}\ e^{-\frac{t\theta_1}{1 + t\gamma_1}}}{(1 + t(\gamma_1 - \gamma_2))^{-\mathfrak{g} - m_1 + m_2}\ e^{-\frac{t(\theta_1 + \theta_2 + \sigma_{12})}{1 + t(\gamma_1 - \gamma_2)}}}\ . \tag{39}$$

3.3.1. *The Grothendieck embedding.* We begin with a simple result which shows that the Kähler class associated with any Grothendieck embedding of $\mathcal{Q}uot^E_{\mathcal{E}_0}$ in a projective space coincides (up to a universal factor) with the Kähler class induced from the moduli space $\mathcal{M}^*_t(E,E_A,A_0)$ via the Kobayashi-Hitchin correspondence for a suitable choice of the parameter t. Therefore our computation will also give the degree of this Quot space with respect to the corresponding Grothendieck embedding.

Fix $x_0\in X$, and let $n\in\mathbb{N}$ be sufficiently large so that $h^1(\mathcal{E}_0(nx_0))=h^1(\mathcal{E}(nx_0))=0$ for every $\mathcal{E}\subset\mathcal{E}_0$ with $\deg(\mathcal{E})=m$. Put

$$V:=H^0(\mathcal{E}_0(nx_0))\ ,\ s:=\deg(E)+r_0(n-\mathfrak{g}+1)\ .$$

We obtain a holomorphic map $\iota_n:\mathcal{Q}uot^E_{\mathcal{E}_0}\to\mathbb{G}r_s(V)$ given by

$$\iota_n(\mathcal{E}\subset\mathcal{E}_0):=H^0(\mathcal{E}(nx_0))\subset V\ .$$

The map ι_n is an embedding for sufficiently large $n\in\mathbb{N}$, hence composing with the Plücker embedding $Pl:\mathbb{G}r_s(V)\to\mathbb{P}(\wedge^s(V))$ we obtain a projective embedding

$$j_n:=Pl\circ\iota_n:\mathcal{Q}uot^E_{\mathcal{E}_0}\to\mathbb{P}(\wedge^s(V))\ .$$

Denote by $\mathcal{U}$ the tautological subbundle of $\mathbb{G}r_s(V)$. We have obvious isomorphisms

$$Pl^*(\mathcal{O}_{\wedge^s(V)}(1))\simeq\det(\mathcal{U})^\vee\ ,\ \iota_n^*(\mathcal{U})\simeq\rho_*(\mathscr{E}(nx_0))\ .$$

Here we used the notation $\mathscr{E}(nx_0):=\mathscr{E}\otimes p_X^*(\mathcal{O}(nx_0))$. Therefore

$$c_1(j_n^*(\mathcal{O}_{\wedge^s(V)}(1)))=-c_1(\rho_*(\mathscr{E}(nx_0))\ .$$

Denote by $\{X\}\in H^2(X,\mathbb{Z})$ the cohomological fundamental class of X and put $\bar{\mathfrak{g}}:=\mathfrak{g}-1$. Since $R^1\rho_*(\mathcal{E}(nx_0))=0$, we obtain using the Grothendieck-Riemann-Roch theorem:

$$c_1(j_n^*(\mathcal{O}_{\wedge^s(V)}(1)))=-c_1(\rho_!(\mathscr{E}(nx_0))=$$

$$-\rho_*\left\{\mathrm{ch}(\mathscr{E})p_X^*(e^{n\{X\}}\cup(1-\bar{\mathfrak{g}}\{X\})\right\}^{(2)}=-\rho_*\left\{\mathrm{ch}(\mathscr{E})p_X^*(1+(n-\bar{\mathfrak{g}})\{X\})\right\}^{(2)}$$

$$=\rho_*\left(-\mathrm{ch}_2(\mathscr{E})-(n-\bar{\mathfrak{g}})c_1(\mathscr{E})\cup p_X^*(\{X\})\right)$$

On the other hand, by formula (25) the class of the Kähler metric ω_t on $\mathcal{Q}uot^E_{\mathcal{E}_0}$ induced from the moduli space $\mathcal{M}_t(E,E_A,A_0)$ via the Kobayashi-Hitchin correspondence (see Remark 3.8) is

$$[\omega_t]=4\pi^2\rho_*\left[-\mathrm{ch}_2(\mathscr{E})-\frac{t\mathrm{Vol}_g(X)}{2\pi}c_1(\mathscr{E})\wedge\{X\}\right]\ .$$

Here g is a Kähler metric on X. This proves:

Proposition 3.16. *The Kähler class $c_1(j_n^*(\mathcal{O}_{\wedge^s(V)}(1)))$ of the Grothendieck embedding $j_n:\mathcal{Q}uot^E_{\mathcal{E}_0}\to\mathbb{P}(\wedge^s(V))$ compares to the Kähler class $[\omega_t]$ induced from the moduli space $\mathcal{M}_t(E,E_A,A_0)$ via the Kobayashi-Hitchin correspondence as follows:*

$$c_1(j_n^*(\mathcal{O}_{\wedge^s(V)}(1)))=\frac{1}{4\pi^2}\left[\omega_{\frac{2\pi(n-\bar{\mathfrak{g}})}{\mathrm{Vol}_g(X)}}\right]\ . \tag{40}$$

3.3.2. *Localization.* Suppose that $\mathcal{E}_0$ decomposes as

$$\mathcal{E}_0 = \oplus_{i=1}^r \mathcal{L}_i \ ,$$

where $\mathcal{L}_i$ are line bundles on X. Putting $l_i := \deg(\mathcal{L}_i)$ we have

$$l := \sum_{i=1}^r l_i = \deg(\mathcal{E}_0) \ .$$

Let

$$I_r(d) := \{\underline{d} = (d_1, \dots, d_r) \in \mathbb{N}^r | \ \sum_{i=1}^r d_i = d\}$$

be the set of weak length r decompositions of d. For every $\underline{d} \in I_r(d)$ define

$$Q^{\underline{d}} := \prod_{i=1}^r \mathcal{Q}uot^{l_i - d_i}_{\mathcal{L}_i} \ .$$

Note that we have an obvious embedding

$$j_{\underline{d}} : Q^{\underline{d}} \to \mathcal{Q}uot^E_{\mathcal{E}_0}$$

defined by

$$j_{\underline{d}}(\mathcal{E}_1, \dots \mathcal{E}_r) := \bigoplus_{i=1}^r \mathcal{E}_i$$

for every system $(\mathcal{E}_1, \dots \mathcal{E}_r)$ of rank 1 subsheaves $\mathcal{E}_i \subset \mathcal{L}_i$ with $\deg(\mathcal{E}_i) = l_i - d_i$.

Remark 3.17. *For every $\underline{d} \in I_r(d)$ there is a canonical isomorphism*

$$\mathscr{E}\big|_{Q^{\underline{d}}} \simeq \bigoplus_{i=1}^r (p_i^{\underline{d}} \times \mathrm{id}_X)^*(\mathscr{E}_i^{\underline{d}}) \ , \tag{41}$$

where $p_i^{\underline{d}} : Q^{\underline{d}} \to \mathcal{Q}uot^{l_i-d_i}_{\mathcal{L}_i}$ denotes the projection onto the i-th factor, and $\mathscr{E}_i^{\underline{d}}$ stands for the universal kernel on $\mathcal{Q}uot^{l_i-d_i}_{\mathcal{L}_i} \times X$.

Endow now the moduli space $\mathcal{Q}uot^E_{\mathcal{E}_0}$ with the $\mathbb{C}^*$-action associated with the morphism $\mathbb{C}^* \to \mathrm{Aut}(\mathcal{E}_0)$ given by $z \mapsto \bigoplus_{i=1}^r z^{w_i} \mathrm{id}_{\mathcal{L}_i}$ (see Theorem 3.6). Note that for every $\underline{d} \in I_r(d)$ one has $\mathcal{Q}^{\underline{d}} \subset [\mathcal{Q}uot^E_{\mathcal{E}_0}]^{\mathbb{C}^*}$ and, endowing the universal kernel $\mathscr{E}_i^{\underline{d}}$ with the $\mathbb{C}^*$-action $z \mapsto z^{w_i} \mathrm{id}_{\mathscr{E}_i^{\underline{d}}}$, the isomorphism (41) becomes an isomorphism of $\mathbb{C}^*$-bundles over the trivial $\mathbb{C}^*$-space $\mathcal{Q}^{\underline{d}}$. We will need the equivariant first Chern class of the right hand summands. Via the standard isomorphism $H^*_{\mathbb{C}^*}(\mathcal{Q}^{\underline{d}}, \mathbb{R}) \simeq H^*(\mathcal{Q}^{\underline{d}}, \mathbb{R})[u]$ we get by (32), (33) formulae of the form:

$$c_1^{\mathbb{C}^*}(\mathscr{E}_i^{\underline{d}}) = -(\gamma_i^{\underline{d}} - w_i u) \otimes 1 + \delta_i^{\underline{d}} + (l_i - d_i) \otimes \{X\} \ , \tag{42}$$

$$c_1^{\mathbb{C}^*}(\mathscr{E}_i^{\underline{d}})^2/[X] = -2\theta_i^{\underline{d}} - 2(l_i - d_i)(\gamma_i^{\underline{d}} - w_i u) \ , \tag{43}$$

where $\gamma_i^{\underline{d}}$, $\delta_i^{\underline{d}}$, $\Theta_i^{\underline{d}}$ are the classes associated with the Quot space $\mathcal{Q}uot^{l_i-d_i}_{\mathcal{L}_i}$ as explained in section 3.2.1. On the other hand, by Theorem 3.6 and Remarks 3.7, 3.8 we know that the class

$$\Omega_t := \left[-\mathrm{ch}_2^{\mathbb{C}^*}(\mathscr{E}) - \frac{t\mathrm{Vol}_g(X)}{2\pi} c_1^{\mathbb{C}^*}(\mathscr{E}) \cup \{X\} \right] / [X]$$

is a lift of $\frac{1}{4\pi^2}[\omega_t]$ in $H^2_{\mathbb{C}^*}(\mathcal{Q}uot^E_{\mathcal{E}_0}, \mathbb{R})$.

Using Remark 3.17 and the additivity of the equivariant Chern character with respect to direct sums we obtain

$$\Omega_t\big|_{Q^{\underline{d}}} = \sum_{i=1}^{r} (p_i^{\underline{d}})^* \left\{ \left[-\frac{1}{2} c_1^{\mathbb{C}^*}(\mathscr{E}_i^{\underline{d}})^2 - \frac{t\mathrm{Vol}_g(X)}{2\pi} c_1^{\mathbb{C}^*}(\mathscr{E}_i^{\underline{d}}) \cup \{X\} \right] / [X] \right\}$$

$$= \sum_{i=1}^{r} [(s_i^{\underline{d}} \gamma_i^{\underline{d}} + \theta_i^{\underline{d}}) - s_i^{\underline{d}} w_i u] \ , \tag{44}$$

where

$$s_i^{\underline{d}} := t\frac{\mathrm{Vol}_g(X)}{2\pi} + (l_i - d_i) \ ,$$

and on the right we have omitted the symbol $(p_i^{\underline{d}})^*$ in front of $(s_i \gamma_i^{\underline{d}} + \theta_i^{\underline{d}})$ to save on notations.

We will also need the equivariant Euler class of the normal bundle $\mathcal{N}_{Q^{\underline{d}}}$ of $Q^{\underline{d}}$ in $\mathcal{Q}uot^E_{\mathcal{E}_0}$. Using the isomorphism (36) together with cohomology and base change we obtain the direct sum decompositions

$$\mathcal{T}_{\mathcal{Q}uot^E_{\mathcal{E}_0}}\big|_{Q^{\underline{d}}} = \bigoplus_{i,j} (p_{ij}^{\underline{d}})^* \mathcal{N}_{ij}^{\underline{d}} \ , \ \mathcal{N}_{Q^{\underline{d}}} = \bigoplus_{i \neq j} (p_{ij}^{\underline{d}})^* \mathcal{N}_{ij}^{\underline{d}} \ . \tag{45}$$

Here $\mathcal{N}_{ij}^{\underline{d}}$ denotes the rank d_j bundle on the product $\mathcal{Q}uot_{\mathcal{L}_i}^{l_i - d_i} \times \mathcal{Q}uot_{\mathcal{L}_j}^{l_j - d_j}$ studied in section 3.2.1, and $p_{ij}^{\underline{d}}$ denotes the projection of $Q^{\underline{d}}$ onto this product. These decompositions are $\mathbb{C}^*$-invariant and $\mathbb{C}^*$ acts with weight $w_j - w_i$ on the summand $\mathcal{N}_{ij}^{\underline{d}}$. Using (39) we obtain

$$e^{\mathbb{C}^*}(\mathcal{N}_{ij}^{\underline{d}}) = ((w_j - w_i)u)^{d_j} c_{\frac{1}{(w_j - w_i)u}}(\mathcal{N}_{ij})$$

$$= ((w_j - w_i)u)^{d_j} \frac{\left[1 + \frac{1}{(w_j - w_i)u}\gamma_i\right]^{-\bar{\mathfrak{g}} - l_i + d_i + l_j} e^{-\frac{\theta_i}{(w_j - w_i)u + \gamma_i}}}{\left[1 + \frac{1}{(w_j - w_i)u}(\gamma_i - \gamma_j)\right]^{-\bar{\mathfrak{g}} - l_i + d_i + l_j - d_j} e^{-\frac{\theta_i + \theta_j + \sigma_{ij}}{(w_j - w_i)u + (\gamma_i - \gamma_j)}}}$$

$$= \frac{((w_j - w_i)u + \gamma_i)^{-\bar{\mathfrak{g}} - l_i + d_i + l_j} \ e^{-\frac{\theta_i}{(w_j - w_i)u + \gamma_i}}}{((w_j - w_i)u + (\gamma_i - \gamma_j))^{-\bar{\mathfrak{g}} - l_i + d_i + l_j - d_j} \ e^{-\frac{\theta_i + \theta_j + \sigma_{ij}}{(w_j - w_i)u + (\gamma_i - \gamma_j)}}} \ .$$

By (45) this gives

$$e^{\mathbb{C}^*}(\mathcal{N}_{Q^{\underline{d}}}) = \prod_{i \neq j} \frac{((w_j - w_i)u + \gamma_i)^{-\bar{\mathfrak{g}} - l_i + d_i + l_j} \ e^{-\frac{\theta_i}{(w_j - w_i)u + \gamma_i}}}{((w_j - w_i)u + (\gamma_i - \gamma_j))^{-\bar{\mathfrak{g}} - l_i + d_i + l_j - d_j} \ e^{-\frac{\theta_i + \theta_j + \sigma_{ij}}{(w_j - w_i)u + (\gamma_i - \gamma_j)}}}$$

$$= (-1)^{\bar{\mathfrak{g}}\binom{r}{2} + (r-1)(l-d)} \frac{\prod_{i \neq j} ((w_j - w_i)u + \gamma_i)^{-\bar{\mathfrak{g}} - l_i + d_i + l_j} \ e^{-\frac{\theta_i}{(w_j - w_i)u + \gamma_i}}}{\prod_{i < j} ((w_j - w_i)u + (\gamma_i - \gamma_j))^{-2\bar{\mathfrak{g}}}} \ .$$

From now on we suppose that the weights w_i are pairwise distinct. This implies

$$\{\mathcal{Q}uot^E_{\mathcal{E}_0}\}^{\mathbb{C}^*} = \coprod_{\underline{d} \in I_r(d)} Q^{\underline{d}} \ ,$$

so that we can use the integration formula ([AB] (3.8)) to compute the volume V_t of $\mathcal{Q}uot^E_{\mathcal{E}_0}$ with respect to the Kähler form $\frac{1}{4\pi^2}\omega_t$:

$$V_t = \frac{1}{(rd)!} \sum_{\underline{d}\in I_r(d)} \left\langle \left\{ \frac{([\Omega_t]\big|_{Q^{\underline{d}}})^{rd}}{e^{\mathbb{C}^*}(\mathcal{N}_{Q^{\underline{d}}})} \right\}, [Q^{\underline{d}}] \right\rangle$$

Here the expression $\frac{(\Omega_t\big|_{Q^{\underline{d}}})^{rd}}{e^{\mathbb{C}^*}(\mathcal{N}_{Q^{\underline{d}}})}$ is regarded as an element in the ring $H^*(Q^{\underline{d}},\mathbb{R})[[u,u^{-1}]]$. The degree with respect to u of the terms with coefficients in $H^{2d}(Q^{\underline{d}},\mathbb{R})$ is 0, hence the formula above yields a real number.

Using our formulae for $[\Omega_t]$ and $e^{\mathbb{C}^*}(\mathcal{N}_{Q^{\underline{d}}})$ we obtain the following formula for the volume V_t:

$$V_t = (-1)^{\bar{\mathfrak{g}}\binom{r}{2}+(r-1)(l-d)} \frac{1}{(rd)!} \sum_{\underline{d}\in I_r(d)}$$

(46)
$$\left\langle \frac{\left(\sum_{i=1}^r [(s^{\underline{d}}_i\gamma^{\underline{d}}_i + \theta^{\underline{d}}_i) - s^{\underline{d}}_i w_i u]\right)^{rd} \prod_{i\neq j}\Big((w_j-w_i)u+\gamma^{\underline{d}}_i\Big)^{\bar{\mathfrak{g}}+l_i-d_i-l_j} e^{\frac{\theta^{\underline{d}}_i}{(w_j-w_i)u+\gamma^{\underline{d}}_i}}}{\prod_{i<j}\Big((w_j-w_i)u+(\gamma^{\underline{d}}_i-\gamma^{\underline{d}}_j)\Big)^{2\bar{\mathfrak{g}}}}, [Q^{\underline{d}}] \right\rangle,$$

where

$$s^{\underline{d}}_i = t\frac{\mathrm{Vol}_g(X)}{2\pi} + l_i - d_i\ .$$

Formula (46) provides an algorithm which generates explicit formulae for V_t:

- For $\underline{d}\in I_r(d)$, $w=(w_1,\dots,w_r)\in\mathbb{C}^r$ (with $w_i\neq w_j$ for $i\neq j$) and $u\in\mathbb{C}^*$ put

$$F^{\underline{d}}_{w,u}(x_1,y_1,\dots,x_r,y_r) :=$$
$$\frac{\left(\sum_{i=1}^r [(s^{\underline{d}}_i x_i + y_i) - s^{\underline{d}}_i w_i u]\right)^{rd} \prod_{i\neq j}((w_j-w_i)u+x_i)^{\bar{\mathfrak{g}}+l_i-d_i-l_j} e^{\frac{y_i}{(w_j-w_i)u+x_i}}}{\prod_{i<j}((w_j-w_i)u+(x_i-x_j))^{2\bar{\mathfrak{g}}}},$$

  and write explicitly the multi-homogeneous part

$$\sum_{\alpha_i+\beta_i=d_i} [F^{\underline{d}}_w]_{\alpha_1,\beta_1\dots,\alpha_r,\beta_r} x_1^{\alpha_1}y_1^{\beta_1}\dots\ x_r^{\alpha_r}y_r^{\beta_r}$$

  of multi-degree $(d_1,\dots,d_r)$ in the Taylor expansion of $F^{\underline{d}}_{w,u}$ at 0 . This expression has degree 0 with respect to u, hence we omitted u in the notation of the coefficients.

- Write

$$V_t = \frac{(-1)^{\bar{\mathfrak{g}}\binom{r}{2}+(r-1)(l-d)}}{(rd)!} \sum_{\underline{d}\in I_r(d)} \sum_{\alpha_i+\beta_i=d_i}$$
$$[F^{\underline{d}}_w]_{\alpha_1,\beta_1,\dots,\alpha_r,\beta_r} \prod_{i=1}^r \mathfrak{g}(\mathfrak{g}-1)\dots(\mathfrak{g}-\beta_i+1)\ .$$

Since the final result is independent of the (pairwise distinct) weights, one can simplify the formula for V_t by choosing the w_i's suitably, e.g., as roots of unity (see for instance [MO]). Our algorithm can be implemented on a computer in order to compute volumina for given data $\mathrm{Vol}_g(X)$ and $(\mathfrak{g}, d, l, r)$.

We have computed the following examples:

Examples: Let $r = 2$ and put $\mu := \mu(\mathcal{E}_0) = \frac{l}{2}$, $\mathfrak{t} = \frac{t\mathrm{Vol}_g(X)}{2\pi}$.

(1) $d = 1$:

$$V_t = \mathfrak{t} + \mu + \bar{\mathfrak{g}} \ .$$

(2) $d = 2$, l even:

$$V_t = \frac{1}{4!}\left\{4(\mathfrak{t} + \mu + \bar{\mathfrak{g}})\big[3(\mathfrak{t} + \mu + \bar{\mathfrak{g}}) - 4\big] - 6\bar{\mathfrak{g}}\right\} \ .$$

For $t_n = \frac{2\pi(n-\bar{\mathfrak{g}})}{\mathrm{Vol}_g(X)}$ (or equivalently $\mathfrak{t}_n := n - \bar{\mathfrak{g}}$) one gets the degree of the image of the Grothendieck embedding $j_n : \mathcal{Q}uot^E_{\mathcal{E}_0} \to \mathbb{P}(\wedge^s(V))$.

(1) $d = 1$:

$$\deg(j_n(\mathcal{Q}uot^E_{\mathcal{E}_0})) = 2n + l \ .$$

(2) $d = 2$, l even:

$$\deg(j_n(\mathcal{Q}uot^E_{\mathcal{E}_0})) = (2n + l)[3(2n + l) - 8] - 6\bar{\mathfrak{g}} \ .$$

We believe it should be possible to obtain a closed formula for V_t as a polynomial of degree d in $\mathfrak{t}$ depending only on $\mathrm{Vol}_g(X)$ and the topological data $(\mathfrak{g}, d, l, r)$. We have partial results in special cases, but not a general formula.

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Christian Okonek:

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland, e-mail: okonek@math.uzh.ch

Andrei Teleman:

Aix Marseille Université, CNRS, Centrale Marseille, I2M, UMR 7373, 13453 Marseille, France, e-mail: andrei.teleman@univ-amu.fr